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THETANULLS OF CYCLIC CURVES OF SMALL GENUS

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ABSTRACT. We study relations among the classical thetanulls of cyclic curves, namely curves $\mathcal{X}$ (of genus $g(\mathcal{X}) > 1$) with an automorphism σ such that σ generates a normal subgroup of the group G of automorphisms, and $g(\mathcal{X}/\langle\sigma\rangle) = 0$. Relations between thetanulls and branch points of the projection are the object of much classical work, especially for hyperelliptic curves, and of recent work, in the cyclic case. We determine the curves of genus 2 and 3 in the locus $\mathcal{M}_g(G, \mathbf{C})$ for all G that have a normal subgroup $\langle\sigma\rangle$ as above, and all possible signatures $\mathbf{C}$, via relations among their thetanulls.

1. INTRODUCTION

In this paper we consider cyclic algebraic curves, over the complex numbers. These are by definition compact Riemann surfaces $\mathcal{X}$ of genus $g > 1$ (unless we allow singular points, as noted below, so as not attach unnecessary qualifications to a definition or statement), admitting an automorphism σ such that $\mathcal{X}/\sigma \cong \mathbb{P}^1$ and σ generates a normal subgroup of the automorphism group $Aut(\mathcal{X})$ of $\mathcal{X}$. When the curve is hyperelliptic, we insist that the curve have "extra automorphisms", in particular σ is not the hyperelliptic involution. Note that the condition implies to having an equation $y^n = f(x)$ for the curve, where x is an affine coordinate on $\mathbb{P}^1$, σ has order n, and $1, y, \sigma y, ..., \sigma^{n-1}y$ is a basis of $\mathbb{C}(\mathcal{X})/\mathbb{C}(x)$. Naturally, the branch points of $\pi : \mathcal{X} \to \mathbb{P}^1$, together with the signature $\mathbf{C}$ of the cover (its monodromy up to conjugation) provide algebraic coordinates for the curve in moduli, though the same curve could be represented in different ways. The problem of expressing these algebraic data in terms of the transcendental (period matrix, thetanulls, e.g.) is classical. We use below formulas for genus-2 curves due to Rosenhein and Picard, Thomae's formulas for hyperelliptic curves, and a recent generalization of the latter for cyclic curves with $\langle s\rangle \cong C_3$, where we denote by C_n the cyclic group of order n, due to Nakayashiki [8]; several other authors recently obtained partial generalizations to cyclic curves also. We do not aim here at a complete account of the classical or contemporary work on these problems.

Cyclic curves are rare in the moduli space $\mathcal{M}_g$ of smooth curves, and it is desirable to characterize their locus, by algebraic conditions on the equation of the curve, or by analytic conditions on its Abelian coordinates, in other words, theta functions, and better yet, by both. We achieve this for genera 2 and 3, making

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recourse to classical formulas, some recent results of Hurwitz space theory, and symbolic manipulation.

The contents of the paper are as follows. In section 2 we recall the notation for Riemann's theta function, as well as classical facts on theta characteristics; we recall Frobenius' and Thomae's formulas for hyperelliptic curves. In sections 3 and 4, respectively, we specialize to the case of genera 2 and 3, we recall recent results on $\mathcal{M}_g(G, \mathbf{C})$, and we calculate thetanull constraints that define the loci of the cyclic curves, using the results we cited. The cleanest case is the one of genus 2 and $\langle\sigma\rangle \cong C_2$, which was classified by Jacobi who gave a condition in terms the branch points of the hyperelliptic involution; such a condition was extended, in principle, to any curve in $\mathcal{M}_g(C_n, \mathbf{C})$, cf. [3] or [9], but the algebraic equation satisfied by the branch points would rapidly become intractable with the size of n.

2. Preliminaries

In this section we give a brief description of the basic setup. All of this material can be found in any standard book on theta functions.

Let $\mathcal{X}$ be a genus $g \geq 2$ algebraic curve. We choose a symplectic homology basis for $\mathcal{X}$, say $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$, such that the intersection products $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. We choose a basis $\{w_i\}$ for the space of holomorphic 1-forms such that $\int_{A_i} w_j = \delta_{ij}$. The matrix $\Omega = \left[\int_{B_i} w_j\right]$ is the *period matrix* of $\mathcal{X}$. The columns of the matrix $[I \,|\Omega]$ form a lattice L in $\mathbb{C}^g$ and the Jacobian of $\mathcal{X}$ is Jac $(\mathcal{X}) = \mathbb{C}^g/L$. Let $\mathcal{H}_g$ be the *Siegel upper-half space*. Then $\Omega \in \mathcal{H}_g$ and there is an injection

$$\mathcal{M}_g \hookrightarrow \mathcal{H}_g/Sp_{2g}(\mathbb{Z}) =: \mathcal{A}_g$$

where $Sp_{2g}(\mathbb{Z})$ is the *symplectic group*. For any $z \in \mathbb{C}^g$ and $\tau \in \mathcal{H}_g$ *Riemann's theta function* is defined as

$$\theta(z, \tau) = \sum_{u \in \mathbb{Z}^g} e^{\pi i(u^t \tau u + 2u^t z)}$$

where u and z are $g-$dimensional column vectors and the products involved in the formula are matrix products. The fact that the imaginary part of τ is positive makes the series absolutely convergent over any compact sets. Therefore, the function is analytic. The theta function is holomorphic on $\mathbb{C}^g \times \mathcal{H}_g$ and satisfies

$$\theta(z + u, \tau) = \theta(z, \tau), \quad \theta(z + u\tau, \tau) = e^{-\pi i(u^t \tau u + 2z^t u)} \cdot \theta(z, \tau),$$

where $u \in \mathbb{Z}^g$; see [6] for details. Any point $e \in$ Jac $(\mathcal{X})$ can be written uniquely as $e = (b, a)\begin{pmatrix} 1_g \\ \Omega \end{pmatrix}$, where $a, b \in \mathbb{R}^g$. We shall use the notation $[e] = \begin{bmatrix} a \\ b \end{bmatrix}$ for the characteristic of e. For any $a, b \in \mathbb{Q}^g$, the theta function with rational characteristics is defined as

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{u \in \mathbb{Z}^g} e^{\pi i((u+a)^t \tau (u+a) + 2(u+a)^t (z+b))}.$$

When the entries of column vectors a and b are from the set $\{0, \frac{1}{2}\}$, then the characteristics $\begin{bmatrix} a \\ b \end{bmatrix}$ are called the *half-integer characteristics*. The corresponding theta functions with rational characteristics are called *theta characteristics*. A scalar obtained by evaluating a theta characteristic at $z = 0$ is called a *theta* *constant*. Points of order n on Jac $\mathcal{X}$ are called the $\frac{1}{n}$-*periods*. Any half-integer characteristic is given by

$$\mathfrak{m} = \frac{1}{2}m = \frac{1}{2}\begin{pmatrix} m_1 & m_2 & \cdots & m_g \\ m_1' & m_2' & \cdots & m_g' \end{pmatrix}$$

where $m_i, m_i' \in \mathbb{Z}$. For $\gamma = \begin{bmatrix} \gamma' \\ \gamma'' \end{bmatrix} \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ we define $e_*(\gamma) = (-1)^{4(\gamma')^t\gamma''}$. Then,

$$\theta[\gamma](-z,\tau) = e_*(\gamma)\theta[\gamma](z,\tau).$$

We say that γ is an **even** (resp. **odd**) characteristic if $e_*(\gamma) = 1$ (resp. $e_*(\gamma) = -1$). For any curve of genus g, there are $2^{g-1}(2^g+1)$ (respectively $2^{g-1}(2^g-1)$) even theta functions (respectively odd theta functions). Let $\mathfrak{a}$ be another half integer characteristic. We define $\mathfrak{m}\,\mathfrak{a}$ as follows.

$$\mathfrak{m}\,\mathfrak{a} = \frac{1}{2}\begin{pmatrix} t_1 & t_2 & \cdots & t_g \\ t_1' & t_2' & \cdots & t_g' \end{pmatrix}$$

where $t_i \equiv (m_i + a_i) \mod 2$ and $t_i' \equiv (m_i' + a_i') \mod 2$.

For the rest of this section we consider only characteristics $\frac{1}{2}q$ in which each of the elements q_i, q_i' is either 0 or 1. We use the following abbreviations

$$|\mathfrak{m}| = \sum_{i=1}^{g} m_i m_i', \qquad |\mathfrak{m},\mathfrak{a}| = \sum_{i=1}^{g}(m_i' a_i - m_i a_i'),$$

$$|\mathfrak{m},\mathfrak{a},\mathfrak{b}| = |\mathfrak{a},\mathfrak{b}| + |\mathfrak{b},\mathfrak{m}| + |\mathfrak{m},\mathfrak{a}|, \qquad \begin{pmatrix}\mathfrak{m}\\ \mathfrak{a}\end{pmatrix} = e^{\pi i \sum_{j=1}^{g} m_j a_j'}.$$

The set of all half integer characteristics forms a group Γ which has 2^{2g} elements. We say that two half integer characteristics $\mathfrak{m}$ and $\mathfrak{a}$ are *syzygetic* (resp., *azygetic*) if $|\mathfrak{m},\mathfrak{a}| \equiv 0 \mod 2$ (resp., $|\mathfrak{m},\mathfrak{a}| \equiv 1 \mod 2$) and three half integer characteristics $\mathfrak{m}, \mathfrak{a}$, and $\mathfrak{b}$ are syzygetic if $|\mathfrak{m},\mathfrak{a},\mathfrak{b}| \equiv 0 \mod 2$.

A *Göpel group* G is a group of 2^r half integer characteristics where $r \le g$ such that every two characteristics are syzygetic. The elements of the group G are formed by the sums of r fundamental characteristics; see [4, pg. 489] for details. Obviously, a Göpel group of order 2^r is isomorphic to C_2^r. The proof of the following lemma can be found on [4, pg. 490].

Lemma 1. *The number of different Göpel groups which have 2^r characteristics is*

$$\frac{(2^{2g}-1)(2^{2g-2}-1)\cdots(2^{2g-2r+2}-1)}{(2^r-1)(2^{r-1}-1)\cdots(2-1)}$$

If G is a Göpel group with 2^r elements, then it has 2^{2g-r} cosets. The cosets are called *Göpel systems* and denoted by $\mathfrak{a}G$, $\mathfrak{a} \in \Gamma$. Any three characteristics of a Göpel system are syzygetic. We can find a set of characteristics called a basis of the Göpel system which derives all its 2^r characteristics by taking only the combinations of any odd number of characteristics of the basis.

Lemma 2. *Let $g \ge 1$ be a fixed integer, r be as defined above and $\sigma = g - r$. Then there are $2^{\sigma-1}(2^\sigma+1)$ Göpel systems which consist of even characteristics only and there are $2^{\sigma-1}(2^\sigma-1)$ Göpel systems which consist of odd characteristics. The other $2^{2\sigma}(2^r-1)$ Göpel systems consist as many odd characteristics as even characteristics.*

Proof. The proof can be found on [4, pg. 492]. ∎

Corollary 3. *When $r = g$ we have only one (resp., 0) Göpel system which consists of even (resp., odd) characteristics.*

Proposition 4. *The following statements are true.*

$$\theta^2[\mathfrak{a}]\theta^2[\mathfrak{a}\mathfrak{h}] = \frac{1}{2^{g-1}} \sum_{\mathfrak{e}} e^{\pi i |\mathfrak{a}\mathfrak{e}|} \binom{\mathfrak{h}}{\mathfrak{a}\mathfrak{e}} \theta^2[\mathfrak{e}]\theta^2[\mathfrak{e}\mathfrak{h}] \tag{1}$$

$$\theta^4[\mathfrak{a}] + e^{\pi i |\mathfrak{a},\mathfrak{h}|}\theta^4[\mathfrak{a}\mathfrak{h}] = \frac{1}{2^{g-1}} \sum_{\mathfrak{e}} e^{\pi i |\mathfrak{a}\mathfrak{e}|} \{\theta^4[\mathfrak{e}] + e^{\pi i |\mathfrak{a},\mathfrak{h}|}\theta^4[\mathfrak{e}\mathfrak{h}]\} \tag{2}$$

where $\theta[e]$ is the theta constant corresponding to the characteristic e, $\mathfrak{a}$ and $\mathfrak{h}$ are any half integer characteristics and $\mathfrak{e}$ is an even characteristic such that $|\mathfrak{e}| \equiv |\mathfrak{e}\mathfrak{h}|$ mod 2*. There are $2 \cdot 2^{g-2}\,(2^{g-1}+1)$ such candidates for $\mathfrak{e}$.*

Proof. For the proof, see [4, pg. 524]. □

The statements given in the proposition above can be used to get identities among theta constants; see section 3.

2.1. **Cyclic curves with extra automorphisms.** A normal cyclic curve is an algebraic curve $\mathcal{X}$ such that there exist a normal cyclic subgroup $C_m \triangleleft \mathrm{Aut}(\mathcal{X})$ such that $g(\mathcal{X}/C_m) = 0$. Then $\bar{G} = G/C_m$ embeds as a finite subgroup of $PGL(2,\mathbb{C})$. An affine equation of a birational model of a cyclic curve can be given by the following

$$y^m = f(x) = \prod_{i=1}^{s} (x-\alpha_i)^{d_i},\ 0 < d_i < m. \tag{3}$$

Hyperelliptic curves are cyclic curves with $m = 2$. Note that when $0 < d_i$ for some i the curve is singular. A hyperelliptic curve $\mathcal{X}$ is a cover of order two of the projective line $\mathbb{P}^1$. Let z be the generator (the hyperelliptic involution) of the Galois group $Gal(\mathcal{X}/\mathbb{P}^1)$. It is known that $\langle z\rangle$ is a normal subgroup of the automorphism group $\mathrm{Aut}(\mathcal{X})$. Let $\mathcal{X} \longrightarrow \mathbb{P}^1$ be the degree 2 hyperelliptic projection. We can assume that infinity is a branch point. Let

$$B := \{\alpha_1, \alpha_2, \cdots, \alpha_{2g+1}\}$$

be the set of other branch points. Let $S = \{1, 2, \cdots, 2g+1\}$ be the index set of B and $\eta : S \longrightarrow \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ be a map defined as follows;

$$\eta(2i-1) = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \cdots & \frac{1}{2} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\eta(2i) = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{bmatrix}$$

where the nonzero element of the first row appears in i^{th} column. We define $\eta(\infty)$ to be $\begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$. For any $T \subset B$, we can define the half-integer characteristic as

$$\eta_T = \sum_{a_k \in T} \eta(k).$$

Let T^c denote the complement of T in B. Note that $\eta_B \in \mathbb{Z}^{2g}$. If we view η_T as an element of $\frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ then $\eta_T = \eta_{T^c}$. Let $\triangle$ denote the symmetric difference of sets, that is $T \bigtriangleup R = (T \cup R) - (T \cap R)$. It can be shown that the set of subsets of B is a group under $\bigtriangleup$. We have the following group isomorphism

$$\{T \subset B \,|\, \#T \equiv g+1 \mod 2\}/T \cong \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}.$$

For hyperelliptic curves, it is known that $2^{g-1}(2^g+1) - \binom{2g+1}{g}$ of the even theta constants are zero. The following theorem provides a condition on the characteristics in which theta characteristics become zero. The proof of the theorem can be found in [7, pg. 102].

Theorem 5. *Let $\mathcal{X}$ be a hyperelliptic curve, with a set B of branch points. Let S be the index set as above and U be the set of all odd values of S. Then for all $T \subset S$ with even cardinality, we have $\theta[\eta_T] = 0$ if and only if $\#(T \triangle U) \neq g+1$, where $\theta[\eta_T]$ is the theta constant corresponding to the characteristics η_T.*

Notice also that by parity, all odd theta constants are zero. There is a formula (so called Frobenius' theta formula) which half-integer theta characteristics for hyperelliptic curves satisfy.

Lemma 6 (Frobenius)**.** *For all $z_i \in \mathbb{C}^g$, $1 \leq i \leq 4$ such that $z_1+z_2+z_3+z_4 = 0$ and for all $b_i \in \mathbb{Q}^{2g}$, $1 \leq i \leq 4$ such that $b_1+b_2+b_3+b_4 = 0$, we have*

$$\sum_{j \in S \cup \{\infty\}} \epsilon_U(j) \prod_{i=1}^{4} \theta[b_i + \eta(j)](z_i) = 0,$$

where for any $A \subset B$,

$$\epsilon_A(k) = \begin{cases} 1 & \text{if } k \in A \\ -1 & \text{otherwise} \end{cases}$$

Proof. See [6, pg. 107].

□

A relationship between theta constants and the branch points of the hyperelliptic curve is given by Thomae's formula.

Lemma 7 (Thomae)**.** *For a non singular even half integer characteristics e corresponding to the partition of the branch points $\{1, 2, \cdots, 2(g+1)\} = \{i_1 < i_2 < \cdots < i_{g+1}\} \cup \{j_1 < j_2 < \cdots < j_{g+1}\}$, we have*

$$\theta[e](0;\tau)^8 = A \prod_{k<l} (\lambda_{i_k} - \lambda_{i_l})^2 (\lambda_{j_k} - \lambda_{j_l})^2.$$

See [6, pg. 128] for the description of A and [6, pg. 120] for the proof. Using Thomae's formula and Frobenius' theta identities we express the branch points of the hyperelliptic curves in terms of even theta constants.

3. Genus 2 curves

The automorphism group G of a genus 2 curve $\mathcal{X}$ in characteristic $\neq 2$ is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_{10}$, V_4, D_8, D_{12}, $SL_2(3)$, $GL_2(3)$, or 2^+S_5. The case when $G \cong 2^+S_5$ occurs only in characteristic 5. If $G \cong SL_2(3)$ (resp., $GL_2(3)$) then $\mathcal{X}$ has equation $Y^2 = X^6 - 1$ (resp., $Y^2 = X(X^4-1)$). If $G \cong \mathbb{Z}_{10}$ then $\mathcal{X}$ has equation $Y^2 = X^6 - X$. For a fixed G from the list above, the locus of genus 2 curves with automorphism group G is an irreducible algebraic subvariety of $\mathcal{M}_2$. Such loci can be described in terms of the Igusa invariants.

For any genus 2 curve we have six odd theta characteristics and ten even theta characteristics. The following are the sixteen theta characteristics, where the first ten are even and the last six are odd. For simplicity, we denote them by $\theta_i = \begin{bmatrix} a \\ b \end{bmatrix}$ instead of $\theta_i \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau)$ where $i = 1, \ldots, 10$ for the even theta functions.

$$\theta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \theta_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \theta_3 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \ \theta_4 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \ \theta_5 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\theta_6 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \ \theta_7 = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \ \theta_8 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \ \theta_9 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{10} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

and the odd theta functions correspond to the following characteristics

$$\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Consider the following Göpel group

$$G = \left\{ 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathfrak{m}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \mathfrak{m}_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \mathfrak{m}_1\mathfrak{m}_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\}.$$

Then, the corresponding Göpel systems are given by:

$$\begin{aligned}
G &= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\} \\
\mathfrak{b}_1 G &= \left\{ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\} \\
\mathfrak{b}_2 G &= \left\{ \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \right\} \\
\mathfrak{b}_3 G &= \left\{ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \right\}
\end{aligned}$$

Notice that from all four cosets, only G has all even characteristics as noticed in Corollary 3. Using the Prop. 4 we have the following six identities for the above Göpel group.

$$\begin{cases}
\theta_5^2\theta_6^2 & = & \theta_1^2\theta_4^2 - \theta_2^2\theta_3^2 \\
\theta_5^4 + \theta_6^4 & = & \theta_1^4 - \theta_2^4 - \theta_3^4 + \theta_4^4 \\
\theta_7^2\theta_9^2 & = & \theta_1^2\theta_3^2 - \theta_2^2\theta_4^2 \\
\theta_7^4 + \theta_9^4 & = & \theta_1^4 - \theta_2^4 + \theta_3^4 - \theta_4^4 \\
\theta_8^2\theta_{10}^2 & = & \theta_1^2\theta_2^2 - \theta_3^2\theta_4^2 \\
\theta_8^4 + \theta_{10}^4 & = & \theta_1^4 + \theta_2^4 - \theta_3^4 - \theta_4^4
\end{cases}$$

These identities express even theta constants in terms of four theta constants. We call them fundamental theta constants θ_1, θ_2, θ_3, θ_4.

Next we find the relation between theta characteristics and branch points of a genus two curve.

Lemma 8 (Picard)**.** *Let a genus 2 curve be given by*

$$Y^2 = X(X-1)(X-\lambda)(X-\mu)(X-\nu). \tag{4}$$

Then, λ, μ, ν can be written as follows:

$$\lambda = \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}, \quad \mu = \frac{\theta_3^2\theta_8^2}{\theta_4^2\theta_{10}^2}, \quad \nu = \frac{\theta_1^2\theta_8^2}{\theta_2^2\theta_{10}^2}. \tag{5}$$

Proof. There are several ways for relating λ, μ, ν to theta constants, depending on the ordering of the branch points of the curve. Let $B = \{\nu, \mu, \lambda, 1, 0\}$ be the branch points of the curves in this order and $U = \{\nu, \lambda, 0\}$ be the set of odd branch points. Using Lemma 7 we have the following set of equations of theta constants and branch points.

$$
\begin{array}{ll}
\theta_1^4 = A\,\nu\lambda(\mu-1)(\nu-\lambda) & \theta_2^4 = A\,\mu(\mu-1)(\nu-\lambda) \\
\theta_3^4 = A\,\mu\lambda(\mu-\lambda)(\nu-\lambda) & \theta_4^4 = A\,\nu(\nu-\lambda)(\mu-\lambda) \\
\theta_5^4 = A\,\lambda\mu(\nu-1)(\nu-\mu) & \theta_6^4 = A\,(\nu-\mu)(\nu-\lambda)(\mu-\lambda) \\
\theta_7^4 = A\,\mu(\nu-1)(\lambda-1)(\nu-\lambda) & \theta_8^4 = A\,\mu\nu(\nu-\mu)(\lambda-1) \\
\theta_9^4 = A\,\nu(\mu-1)(\lambda-1)(\mu-\lambda) & \theta_{10}^4 = A\,\lambda(\lambda-1)(\nu-\mu),
\end{array}
\tag{6}
$$

where A is a constant. Choosing the appropriate equation from the set Eq. (6) we have the following:

$$
\lambda^2 = \left(\frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}\right)^2 \quad \mu^2 = \left(\frac{\theta_3^2\theta_8^2}{\theta_4^2\theta_{10}^2}\right)^2 \quad \nu^2 = \left(\frac{\theta_1^2\theta_8^2}{\theta_2^2\theta_{10}^2}\right)^2.
$$

Each value for (λ, μ, ν) gives isomorphic genus 2 curves. Hence, we can choose

$$
\lambda = \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}, \quad \mu = \frac{\theta_3^2\theta_8^2}{\theta_4^2\theta_{10}^2}, \quad \nu = \frac{\theta_1^2\theta_8^2}{\theta_2^2\theta_{10}^2}.
$$

This completes the proof. $\square$

One of the main goals of this paper is to describe each locus of genus 2 curves with fixed automorphism group in terms of the fundamental theta constants. We have the following

Corollary 9. *Every genus two curve can be written in the form:*

$$
y^2 = x\,(x-1)\left(x - \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}\right)\left(x^2 - \frac{\theta_2^2\,\theta_3^2 + \theta_1^2\,\theta_4^2}{\theta_2^2\,\theta_4^2}\cdot\alpha\,x + \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}\,\alpha^2\right),
$$

where $\alpha = \frac{\theta_8^2}{\theta_{10}^2}$ *and in terms of* $\theta_1, \dots, \theta_4$ *is given by*

$$
\alpha^2 + \frac{\theta_1^4 + \theta_2^4 - \theta_3^4 - \theta_4^4}{\theta_1^2\theta_2^2 - \theta_3^2\theta_4^2}\,\alpha + 1 = 0
$$

Furthermore, if $\alpha = \pm 1$ *then* $V_4 \hookrightarrow Aut(\mathcal{X})$.

Proof. Let's write the genus 2 curve in the following form:

$$
Y^2 = X(X-1)(X-\lambda)(X-\mu)(X-\nu)
$$

where λ, μ, ν are given by Eq. (5). Let $\alpha := \frac{\theta_8^2}{\theta_{10}^2}$. Then,

$$
\mu = \frac{\theta_3^2}{\theta_4^2}\,\alpha, \quad \nu = \frac{\theta_1^2}{\theta_2^2}\,\alpha
$$

Using the following two identities,

$$
\begin{aligned}
\theta_8^4 + \theta_{10}^4 &= \theta_1^4 + \theta_2^4 - \theta_3^4 - \theta_4^4 \\
\theta_8^2\theta_{10}^2 &= \theta_1^2\theta_2^2 - \theta_3^2\theta_4^2
\end{aligned}
\tag{7}
$$

we have,

$$
\alpha^2 + \frac{\theta_1^4 + \theta_2^4 - \theta_3^4 - \theta_4^4}{\theta_1^2\theta_2^2 - \theta_3^2\theta_4^2}\,\alpha + 1 = 0
\tag{8}
$$

If $\alpha = \pm 1$ the $\mu\nu = \lambda$. It is well known that this implies that the genus 2 curve has an elliptic involution. Hence, $V_4 \hookrightarrow Aut(\mathcal{X})$. $\square$

Remark 10. *i) From the above we have that $\theta_8^4 = \theta_{10}^4$ implies that $V_4 \hookrightarrow Aut(\mathcal{X})$. Lemma 15 determines a necessary and equivalent statement when $V_4 \hookrightarrow Aut(\mathcal{X})$.*

ii) The last part of the lemma above shows that if $\theta_8^4 = \theta_{10}^4$ then all coefficients of the genus 2 curve are given as rational functions of the 4 fundamental theta functions. Such fundamental theta functions determine the field of moduli of the given curve. Hence, the curve is defined over its field of moduli.

Corollary 11. *Let $\mathcal{X}$ be a genus 2 curve which has an elliptic involution. Then $\mathcal{X}$ is defined over its field of moduli.*

This was the main result of [1].

3.1. **Describing the locus of genus two curves with fixed automorphism group by theta constants.** The locus $\mathcal{L}_2$ of genus 2 curves $\mathcal{X}$ which have an elliptic involution is a closed subvariety of $\mathcal{M}_2$. Let $W = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}$ be the set of roots of the binary sextic and A and B be subsets of W such that $W = A \cup B$ and $|A \cap B| = 2$. We define the cross ratio of the two pairs $z_1, z_2; z_3, z_4$ by

$$(z_1, z_2; z_3, z_4) = \frac{z_1; z_3, z_4}{z_2; z_3, z_4} = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4}.$$

Take $A = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ and $B = \{\gamma_1, \gamma_2, \beta_1, \beta_2\}$. Jacobi [2] gives a description of $\mathcal{L}_2$ in terms of the cross ratios of the elements of W.

$$\frac{\alpha_1 - \beta_1}{\alpha_1 - \beta_2} : \frac{\alpha_2 - \beta_1}{\alpha_2 - \beta_2} = \frac{\gamma_1 - \beta_1}{\gamma_1 - \beta_2} : \frac{\gamma_2 - \beta_1}{\gamma_2 - \beta_2}$$

We recall that the following identities hold for cross ratios:

$$(\alpha_1, \alpha_2\,; \beta_1, \beta_2) = (\alpha_2, \alpha_1; \beta_2, \beta_1) = (\beta_1, \beta_2; \alpha_1, \alpha_2) = (\beta_2, \beta_1; \alpha_2, \alpha_1)$$

and

$$(\alpha_1, \alpha_2; \infty, \beta_2) = (\infty, \beta_2; \alpha_1, \alpha_2) = (\beta_2; \alpha_2, \alpha_1)$$

Next we want to use this result to determine relations among theta functions for a genus 2 curve in the locus $\mathcal{L}_2$. Let $\mathcal{X}$ be any genus 2 curve given by equation

$$Y^2 = X(X-1)(X-a_1)(X-a_2)(X-a_3)$$

We take $\infty \in A \cap B$. Then there are five cases for $\alpha \in A \cap B$, where α is an element of the set$\{0, 1, a_1, a_2, a_3\}$. For each of these cases there are three possible relationships for cross ratios as described below:

i) $A \cap B = \{0, \infty\}$: The possible cross ratios are

$$(a_1, 1; \infty, 0) = (a_3, a_2; \infty, 0)$$
$$(a_2, 1; \infty, 0) = (a_1, a_3; \infty, 0)$$
$$(a_1, 1; \infty, 0) = (a_2, a_3; \infty, 0)$$

ii) $A \cap B = \{1, \infty\}$: The possible cross ratios are

$$(a_1, 0; \infty, 1) = (a_2, a_3; \infty, 1)$$
$$(a_1, 0; \infty, 1) = (a_3, a_2; \infty, 1)$$
$$(a_2, 0; \infty, 1) = (a_1, a_3; \infty, 1)$$

iii) $A \cap B = \{a_1, \infty\}$: The possible cross ratios are

$$(1, 0; \infty, a_1) = (a_3, a_2; \infty, a_1)$$
$$(a_2, 0; \infty, a_1) = (1, a_3; \infty, a_1)$$
$$(1, 0; \infty, a_1) = (a_2, a_3; \infty, a_1)$$

iv) $A \cap B = \{a_2, \infty\}$: The possible cross ratios are

$$(1, 0; \infty, a_2) = (a_1, a_3; \infty, a_2)$$
$$(1, 0; \infty, a_2) = (a_3, a_1; \infty, a_2)$$
$$(a_1, 0; \infty, a_2) = (1, a_3; \infty, a_2)$$

v) $A \cap B = \{a_3, \infty\}$: The possible cross ratios are

$$(a_1, 0; \infty, a_3) = (1, a_2; \infty, a_3)$$
$$(1, 0; \infty, a_3) = (a_2, a_1; \infty, a_3)$$
$$(1, 0; \infty, a_3) = (a_1, a_2; \infty, a_3)$$

We summarize these relationships in the following table:

	Cross ratio	$f(a_1, a_2, a_3) = 0$	theta constants
1	$(1, 0; \infty, a_1) = (a_3, a_2; \infty, a_1)$	$a_1a_2 + a_1 - a_3a_1 - a_2$	$-\theta_1^2\theta_3^2\theta_8^2\theta_2^2 - \theta_1^2\theta_2^2\theta_4^2\theta_{10}^2 + \theta_1^4\theta_3^2\theta_{10}^2 + \theta_3^2\theta_2^4\theta_{10}^2$
2	$(a_2, 0; \infty, a_1) = (1, a_3; \infty, a_1)$	$a_1a_2 - a_1 + a_3a_1 - a_3a_2$	$\theta_3^2\theta_8^2\theta_2^2\theta_4^2 - \theta_2^2\theta_4^4\theta_{10}^2 + \theta_1^2\theta_3^2\theta_4^2\theta_{10}^2 - \theta_3^4\theta_2^2\theta_{10}^2$
3	$(1, 0; \infty, a_1) = (a_2, a_3; \infty, a_1)$	$a_1a_2 - a_1 - a_3a_1 + a_3$	$-\theta_8^4\theta_3^2\theta_2^2 + \theta_8^2\theta_2^2\theta_{10}^2\theta_4^2 + \theta_1^2\theta_3^2\theta_8^2\theta_{10}^2 - \theta_3^2\theta_2^2\theta_{10}^4$
4	$(1, 0; \infty, a_2) = (a_1, a_3; \infty, a_2)$	$a_1a_2 - a_2 - a_3a_2 + a_3$	$-\theta_1^2\theta_8^4\theta_4^2 - \theta_1^2\theta_{10}^4\theta_4^2 + \theta_8^2\theta_2^2\theta_{10}^2\theta_4^2 + \theta_1^2\theta_3^2\theta_8^2\theta_{10}^2$
5	$(1, 0; \infty, a_2) = (a_3, a_1; \infty, a_2)$	$a_1a_2 - a_1 + a_2 - a_3a_2$	$-\theta_1^2\theta_8^2\theta_3^2\theta_4^2 + \theta_1^2\theta_{10}^2\theta_4^4 + \theta_1^2\theta_3^4\theta_{10}^2 - \theta_3^2\theta_2^2\theta_{10}^2\theta_4^2$
6	$(a_1, 0; \infty, a_2) = (1, a_3; \infty, a_2)$	$a_1a_2 - a_3a_1 - a_2 + a_3a_2$	$-\theta_1^2\theta_8^2\theta_2^2\theta_4^2 + \theta_1^4\theta_{10}^2\theta_4^2 - \theta_1^2\theta_3^2\theta_2^2\theta_{10}^2 + \theta_2^4\theta_4^2\theta_{10}^2$
7	$(a_1, 0; \infty, a_3) = (1, a_2; \infty, a_3)$	$a_1a_2 - a_3a_1 - a_3a_2 + a_3$	$-\theta_8^4\theta_2^2\theta_4^2 + \theta_1^2\theta_8^2\theta_{10}^2\theta_4^2 - \theta_2^2\theta_{10}^4\theta_4^2 + \theta_3^2\theta_8^2\theta_2^2\theta_{10}^2$
8	$(1, 0; \infty, a_3) = (a_2, a_1; \infty, a_3)$	$a_3a_1 - a_1 - a_3a_2 + a_3$	$\theta_8^4 - \theta_{10}^4$
9	$(1, 0; \infty, a_3) = (a_1, a_2; \infty, a_3)$	$a_3a_1 + a_2 - a_3 - a_3a_2$	$\theta_1^4\theta_8^2\theta_4^2 - \theta_1^2\theta_2^2\theta_4^2\theta_{10}^2 - \theta_1^2\theta_3^2\theta_8^2\theta_2^2 + \theta_8^2\theta_2^4\theta_4^2$
10	$(a_1, 0; \infty, 1) = (a_2, a_3; \infty, 1)$	$-a_1 + a_3a_1 + a_2 - a_3$	$\theta_1^4\theta_3^2\theta_8^2 - \theta_1^2\theta_8^2\theta_2^2\theta_4^2 - \theta_1^2\theta_3^2\theta_2^2\theta_{10}^2 + \theta_3^2\theta_8^2\theta_2^4$
11	$(a_1, 0; \infty, 1) = (a_3, a_2; \infty, 1)$	$a_1a_2 - a_1 - a_2 + a_3$	$\theta_1^2\theta_8^4\theta_3^2 - \theta_1^2\theta_8^2\theta_{10}^2\theta_4^2 + \theta_1^2\theta_3^2\theta_{10}^4 - \theta_3^2\theta_8^2\theta_2^2\theta_{10}^2$
12	$(a_2, 0; \infty, 1) = (a_1, a_3; \infty, 1)$	$a_1 - a_2 + a_3a_2 - a_3$	$\theta_1^2\theta_8^2\theta_4^4 - \theta_1^2\theta_3^2\theta_4^2\theta_{10}^2 + \theta_1^2\theta_3^4\theta_8^2 - \theta_3^2\theta_8^2\theta_2^2\theta_4^2$
13	$(a_1, 1; \infty, 0) = (a_3, a_2; \infty, 0)$	$a_1a_2 - a_3$	$\theta_8^4 - \theta_{10}^4$
14	$(a_2, 1; \infty, 0) = (a_1, a_3; \infty, 0)$	$a_1 - a_3a_2$	$\theta_3^4 - \theta_4^4$
15	$(a_1, 1; \infty, 0) = (a_2, a_3; \infty, 0)$	$a_3a_1 - a_2$	$\theta_1^4 - \theta_2^4$

TABLE 1. Relation of theta functions and cross ratios

Lemma 12. *Let $\mathcal{X}$ be a genus 2 curve. Then $Aut(\mathcal{X}) \cong V_4$ if and only if the theta functions of $\mathcal{X}$ satisfy*

$$
\begin{aligned}
&(\theta_1^4-\theta_2^4)(\theta_3^4-\theta_4^4)(\theta_8^4-\theta_{10}^4)(-\theta_1^2\theta_3^2\theta_8^2\theta_2^2-\theta_1^2\theta_2^2\theta_4^2\theta_{10}^2+\theta_1^4\theta_3^2\theta_{10}^2+\theta_3^2\theta_2^4\theta_{10}^2)\\
&(\theta_3^2\theta_8^2\theta_2^2\theta_4^2-\theta_2^2\theta_4^4\theta_{10}^2+\theta_1^2\theta_3^2\theta_4^2\theta_{10}^2-\theta_3^4\theta_2^2\theta_{10}^2)(-\theta_8^4\theta_3^2\theta_2^2+\theta_8^2\theta_2^2\theta_{10}^2\theta_4^2+\theta_1^2\theta_3^2\theta_8^2\theta_{10}^2-\theta_3^2\theta_2^2\theta_{10}^4)\\
&(-\theta_1^2\theta_8^4\theta_4^2-\theta_1^2\theta_{10}^4\theta_4^2+\theta_8^2\theta_2^2\theta_{10}^2\theta_4^2+\theta_1^2\theta_3^2\theta_8^2\theta_{10}^2)(-\theta_1^2\theta_8^2\theta_3^2\theta_4^2+\theta_1^2\theta_{10}^2\theta_4^4+\theta_1^2\theta_3^4\theta_{10}^2-\theta_3^2\theta_2^2\theta_{10}^2\theta_4^2)\\
&(-\theta_1^2\theta_8^2\theta_2^2\theta_4^2+\theta_1^4\theta_{10}^2\theta_4^2-\theta_1^2\theta_3^2\theta_2^2\theta_{10}^2+\theta_2^4\theta_4^2\theta_{10}^2)(-\theta_8^4\theta_2^2\theta_4^2+\theta_1^2\theta_8^2\theta_{10}^2\theta_4^2-\theta_2^2\theta_{10}^4\theta_4^2+\theta_3^2\theta_8^2\theta_2^2\theta_{10}^2)\\
&(\theta_1^4\theta_8^2\theta_4^2-\theta_1^2\theta_2^2\theta_4^2\theta_{10}^2-\theta_1^2\theta_3^2\theta_8^2\theta_2^2+\theta_8^2\theta_2^4\theta_4^2)(\theta_1^4\theta_3^2\theta_8^2-\theta_1^2\theta_8^2\theta_2^2\theta_4^2-\theta_1^2\theta_3^2\theta_2^2\theta_{10}^2+\theta_3^2\theta_8^2\theta_2^4)\\
&(\theta_1^2\theta_8^4\theta_3^2-\theta_1^2\theta_8^2\theta_{10}^2\theta_4^2+\theta_1^2\theta_3^2\theta_{10}^4-\theta_3^2\theta_8^2\theta_2^2\theta_{10}^2)(\theta_1^2\theta_8^2\theta_4^4-\theta_1^2\theta_3^2\theta_4^2\theta_{10}^2+\theta_1^2\theta_3^4\theta_8^2-\theta_3^2\theta_8^2\theta_2^2\theta_4^2)=0
\end{aligned}
\tag{9}
$$

However, we are unable to get a similar result for cases D_8 or D_{12} by this argument. Instead, we will use the invariants of genus 2 curves and a more computational approach. In the process, we will offer a different proof of the lemma above.

Lemma 13. *i) The locus $\mathcal{L}_2$ of genus 2 curves $\mathcal{X}$ which have a degree 2 elliptic subcover is a closed subvariety of $\mathcal{M}_2$. The equation of $\mathcal{L}_2$ is given by*

$$
\begin{aligned}
&8748J_{10}J_2^4J_6^2-507384000J_{10}^2J_4^2J_2-19245600J_{10}^2J_4J_2^3-592272J_{10}J_4^4J_2^2+77436J_{10}J_4^3J_2^4\\
&-81J_2^3J_6^4-3499200J_{10}J_2J_6^3+4743360J_{10}J_4^3J_2J_6-870912J_{10}J_4^2J_2^3J_6+3090960J_{10}J_4J_2^2J_6^2\\
&-78J_2^5J_4^5-125971200000J_{10}^3+384J_4^6J_6+41472J_{10}J_4^5+159J_4^6J_2^3-236196J_{10}^2J_2^5-80J_4^7J_2\\
&-47952J_2J_4J_6^4+104976000J_{10}^2J_2^2J_6-1728J_4^5J_2^2J_6+6048J_4^4J_2J_6^2-9331200J_{10}J_4^2J_6^2\\
&+12J_2^6J_4^3J_6+29376J_2^2J_4^2J_6^3-8910J_2^3J_4^3J_6^2-2099520000J_{10}^2J_4J_6+31104J_6^5-6912J_4^3J_6^34\\
&-J_2^7J_4^4-5832J_{10}J_2^5J_4J_6-54J_2^5J_4^2J_6^2+108J_2^4J_4J_6^3+972J_{10}J_2^6J_4^2+1332J_2^4J_4^4J_6=0
\end{aligned}
\tag{10}
$$

ii) The locus of genus 2 curves $\mathcal{X}$ with $Aut(\mathcal{X}) \cong D_8$ is given by the equation of $\mathcal{L}_2$ and

$$
1706J_4^2J_2^2+2560J_4^3+27J_4J_2^4-81J_2^3J_6-14880J_2J_4J_6+28800J_6^2=0 \tag{11}
$$

iii) The locus of genus 2 curves $\mathcal{X}$ with $Aut(\mathcal{X}) \cong D_{12}$ is

$$
\begin{aligned}
-J_4J_2^4+12J_2^3J_6-52J_4^2J_2^2+80J_4^3+960J_2J_4J_6-3600J_6^2=0\\
864J_{10}J_2^5+3456000J_{10}J_4^2J_2-43200J_{10}J_4J_2^3-2332800000J_{10}^2-J_4^2J_2^6\\
-768J_4^4J_2^2+48J_4^3J_2^4+4096J_4^5=0
\end{aligned}
\tag{12}
$$

Our goal is to express each of the above loci in terms of the theta characteristics. We obtain the following result.

Theorem 14. *Let $\mathcal{X}$ be a genus 2 curve. Then the following hold:*
i) $Aut(\mathcal{X}) \cong V_4$ if and only if the relations of theta functions given Eq. (9) *holds.*
ii) $Aut(\mathcal{X}) \cong D_8$ if and only if Eq. (1) in [10] *is satisfied.*
iii) $Aut(\mathcal{X}) \cong D_{12}$ if and only if Eq. (2) in [10] *is satisfied.*

Proof. Part i) of the theorem is Lemma 12. Here we give a somewhat different proof. Assume that $\mathcal{X}$ is a genus 2 curve with equation

$$
Y^2 = X(X-1)(X-a_1)(X-a_2)(X-a_3)
$$

whose classical invariants satisfy Eq. (10). Expressing the classical invariants of $\mathcal{X}$ in terms of a_1, a_2, a_3, substituting them into (10), and factoring the resulting equation yields

$$
\begin{aligned}
&(a_1a_2-a_2-a_3a_2+a_3)^2(a_1a_2-a_1+a_3a_1-a_3a_2)^2(a_1a_2-a_3a_1-a_3a_2+a_3)^2\\
&(a_3a_1-a_1-a_3a_2+a_3)^2(a_1a_2+a_1-a_3a_1-a_2)^2(a_1a_2-a_1-a_3a_1+a_3)^2\\
&(a_3a_1+a_2-a_3-a_3a_2)^2(-a_1+a_3a_1+a_2-a_3)^2(a_1a_2-a_1-a_2+a_3)^2\\
&(a_1a_2-a_1+a_2-a_3a_2)^2(a_1-a_2+a_3a_2-a_3)^2(a_1a_2-a_3a_1-a_2+a_3a_2)^2\\
&(a_1a_2-a_3)^2(a_1-a_3a_2)^2(a_3a_1-a_2)^2=0
\end{aligned}
\tag{13}
$$

It is no surprise that we get the 15 factors of Table 1. The relations of theta constants follow from the table. ii) Let $\mathcal{X}$ be a genus 2 curve which has an elliptic involution. Then $\mathcal{X}$ is isomorphic to a curve with equation

$$Y^2 = X(X-1)(X-a_1)(X-a_2)(X-a_1a_2).$$

If $\mathrm{Aut}(\mathcal{X}) \cong D_8$ then the $SL_2(k)$-invariants of such curve must satisfy Eq. (11). Then, we get the equation in terms of a_1, a_2. By writing the relation $a_3 = a_1a_2$ in terms of theta constants, we get $\theta_4^4 = \theta_3^4$. All the results above lead to part ii) of the theorem. iii) The proof of this part is similar to part ii). $\square$

We would like to express the conditions of the previous lemma in terms of the fundamental theta constants only.

Lemma 15. *Let $\mathcal{X}$ be a genus 2 curve. Then we have the following:*

i): $V_4 \hookrightarrow Aut(\mathcal{X})$ *if and only if the fundamental theta constants of $\mathcal{X}$ satisfy*

$$
\begin{aligned}
&\left(\theta_3^4-\theta_4^4\right)\left(\theta_1^4-\theta_3^4\right)\left(\theta_2^4-\theta_4^4\right)\left(\theta_1^4-\theta_4^4\right)\left(\theta_3^4-\theta_2^4\right)\left(\theta_1^4-\theta_2^4\right)\\
&\left(-\theta_4^2+\theta_3^2+\theta_1^2-\theta_2^2\right)\left(\theta_4^2-\theta_3^2+\theta_1^2-\theta_2^2\right)\left(-\theta_4^2-\theta_3^2+\theta_2^2+\theta_1^2\right)\left(\theta_4^2+\theta_3^2+\theta_2^2+\theta_1^2\right)\\
&\left({\theta_1}^4{\theta_2}^4+{\theta_3}^4{\theta_2}^4+{\theta_1}^4{\theta_3}^4-2\,\theta_1^2\theta_2^2\theta_3^2\theta_4^2\right)\left(-{\theta_3}^4{\theta_2}^4-{\theta_2}^4{\theta_4}^4-{\theta_3}^4{\theta_4}^4+2\,\theta_1^2\theta_2^2\theta_3^2\theta_4^2\right)\\
&\left({\theta_2}^4{\theta_4}^4+{\theta_1}^4{\theta_2}^4+{\theta_1}^4{\theta_4}^4-2\,\theta_1^2\theta_2^2\theta_3^2\theta_4^2\right)\left({\theta_1}^4{\theta_4}^4+{\theta_3}^4{\theta_4}^4+{\theta_1}^4{\theta_3}^4-2\,\theta_1^2\theta_2^2\theta_3^2\theta_4^2\right)=0
\end{aligned}
\tag{14}
$$

ii: $D_8 \hookrightarrow Aut(\mathcal{X})$ *if and only if the fundamental theta constants of $\mathcal{X}$ satisfy Eq. (3) in* [10]

iii: $D_6 \hookrightarrow Aut(\mathcal{X})$ *if and only if the fundamental theta constants of $\mathcal{X}$ satisfy Eq. (4) in* [10]

Proof. Notice that Eq. (9) contains only $\theta_1, \theta_2, \theta_3, \theta_4, \theta_8$ and θ_{10}. Using Eq. (7), we can eliminate θ_8 and θ_{10} from Eq. (9). The J_{10} invariant of any genus two curve is given by the following in terms of theta constants:

$$J_{10} = \frac{\theta_1^{12}\theta_3^{12}}{\theta_2^{28}\theta_4^{28}\theta_{10}^{40}}\,(\theta_1^2\theta_2^2-\theta_3^2\theta_4^2)^{12}(\theta_1^2\theta_4^2-\theta_2^2\theta_3^2)^{12}(\theta_1^2\theta_3^2-\theta_2^2\theta_4^2)^{12}.$$

Since $J_{10} \neq 0$ we can cancel the factors $(\theta_1^2\theta_2^2-\theta_3^2\theta_4^2), (\theta_1^2\theta_4^2-\theta_2^2\theta_3^2)$ and $(\theta_1^2\theta_3^2-\theta_2^2\theta_4^2)$ from the equation of V_4 locus. The result follows from Theorem 14. The proof of part ii) and iii) is similar and we avoid details. $\square$

Remark 16. *i) For the other two loci, we can also obtain equations in terms of the fundamental theta constants. However, such equations are big and we don't display them here.*

ii) By using Frobenius's relations we get

$$J_{10} = \frac{(\theta_1\theta_3)^{12}}{(\theta_2\theta_4)^{28}\,\theta_{10}^{16}}\,(\theta_5\theta_6\theta_7\theta_8\theta_9)^{24}$$

Hence, $\theta_i \neq 0$ *for* $i = 1, 3, 5, \dots 9$.

4. Genus 3 cyclic curves

For genus 3 we have hyperelliptic and non-hyperelliptic algebraic curves. The following table gives all possible genus 3 cyclic algebraic curves; see [5] for details. The first 11 cases are for the hyperelliptic curves and the last 12 cases are for the non-hyperelliptic curves.

	$\mathrm{Aut}(\mathcal{X}_g)$	equation	Id.
1	$\mathbb{Z}_2$	$y^2 = x(x-1)(x^5 + ax^4 + bx^3 + cx^2 + dx + e)$	$(2,1)$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$y^2 = x^8 + a_3x^6 + a_2x^4 + a_1x^2 + 1$	$(4,2)$
3	$\mathbb{Z}_4$	$y^2 = x(x^2-1)(x^4 + ax^2 + b)$	$(4,1)$
4	$\mathbb{Z}_{14}$	$y^2 = x^7 - 1$	$(14,2)$
5	$\mathbb{Z}_2^3$	$y^2 = (x^4 + ax^2 + 1)(x^4 + bx^2 + 1)$	$(8,5)$
6	$\mathbb{Z}_2 \times D_8$	$y^2 = x^8 + ax^4 + 1$	$(16,11)$
7	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$y^2 = (x^4 - 1)(x^4 + ax^2 + 1)$	$(8,2)$
8	D_{12}	$y^2 = x(x^6 + ax^3 + 1)$	$(12,4)$
9	U_6	$y^2 = x(x^6 - 1)$	$(24,5)$
10	V_8	$y^2 = x^8 - 1$	$(32,9)$
11	$\mathbb{Z}_2 \times S_4$	$y^2 = x^8 + 14x^2 + 1$	$(48,48)$
12	V_4	$x^4 + y^4 + ax^2y^2 + bx^2 + cy^2 + 1 = 0$	(4,2)
13	D_8	take $b = c$	(8,3)
14	S_4	take $a = b = c$	(24,12)
15	$C_4^2 \rtimes S_3$	take $a = b = c = 0$ or $y^4 = x(x^2-1)$	(96,64)
16	16	$y^4 = x(x-1)(x-t)$	(16,13)
17	48	$y^4 = x^3 - 1$	(48,33)
18	C_3	$y^3 = x(x-1)(x-s)(x-t)$	(3,1)
19	C_6	take $s = 1 - t$	(6,2)
20	C_9	$y^3 = x(x^3 - 1)$	(9,1)
21	$L_3(2)$	$x^3y + y^3z + z^3x = 0$	(168,42)
22	S_3	$a(x^4 + y^4 + z^4) + b(x^2y^2 + x^2z^2 + y^2z^2) +$ $c(x^2yz + y^2xz + z^2xy) = 0$	(6,1)
23	C_2	$x^4 + x^2(y^2 + az^2) + by^4 + cy^3z + dy^2z^2$ $+ eyz^3 + gz^4 = 0$, either $e = 1$ or $g = 1$	(2,1)

Table 2. The list of automorphism groups of genus 3 and their equations

4.1. **Theta functions for hyperelliptic curves.** For genus three hyperelliptic curve we have 28 odd theta characteristics and 36 even theta characteristics. The following shows the corresponding characteristics for each theta function. The first 36 are for the even functions and the last 28 are for the odd functions. For simplicity, we denote them by $\theta_i = \begin{bmatrix} a \\ b \end{bmatrix}$ instead of $\theta_i \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau)$.

$$\theta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \theta_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \ \theta_4 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix},$$

$$\theta_5 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_6 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_7 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \ \theta_8 = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix},$$

$$\theta_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{10} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \theta_{11} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{12} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\theta_{13} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{14} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \theta_{15} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{16} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix},$$

$$\theta_{17} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{18} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \ \theta_{19} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{20} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$\theta_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{22} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \ \theta_{23} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{24} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\theta_{25} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{26} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{27} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \ \theta_{28} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

$$\theta_{29} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \ \theta_{30} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{31} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{32} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix},$$

$$\theta_{33} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{34} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{35} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{36} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\theta_{37} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \ \theta_{38} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{39} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{40} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

$$\theta_{41} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{42} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{43} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \ \theta_{44} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix},$$

$$\theta_{45} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{46} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{47} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{48} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\theta_{49} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{50} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{51} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{52} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\theta_{53} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{54} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{55} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{56} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\theta_{57} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \ \theta_{58} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{59} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \ \theta_{60} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\theta_{61} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \theta_{62} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \ \theta_{63} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \ \theta_{64} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It can be shown that one of the corresponding even theta constants is zero. Let's pick $S = \{1, 2, 3, 4, 5, 6, 7\}$ and $U = \{1, 3, 5, 7\}$. Let $T = U$. Then, by Theorem 5 the theta constant corresponding to the characteristic $\eta_T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ is zero. That is $\theta_{12}(0) = 0$. Next, we give the relation between theta characteristics and branch points of the genus three hyperelliptic curve. Let $B = \{a_1, a_2, a_3, a_4, a_5, 1, 0\}$ be the finite branch points of the curves and $U = \{a_1, a_3, a_5, 0\}$ be the set of odd branch points.

Lemma 17. *Any genus 3 hyperelliptic curve is isomorphic to a curve given by the equation*

$$Y^2 = X(X-1)(X-a_1)(X-a_2)(X-a_3)(X-a_4)(X-a_5),$$

where

$$a_1 = \frac{\theta_{31}^2\theta_{21}^2}{\theta_{34}^2\theta_{24}^2},\ a_2 = \frac{\theta_{31}^2\theta_{13}^2}{\theta_9^2\theta_{24}^2},\ a_3 = \frac{\theta_{11}^2\theta_{31}^2}{\theta_{24}^2\theta_6^2},\ a_4 = \frac{\theta_{21}^2\theta_7^2}{\theta_{15}^2\theta_{34}^2},\ a_5 = \frac{\theta_{13}^2\theta_1^2}{\theta_{26}^2\theta_9^2}.$$

Proof. By using Lemma 7 we have the following set of equation of theta constants and branch points which are ordered $a_1, a_2, a_3, a_4, a_5, 0, 1, \infty$. We use the notation (i,j) for $(a_i - a_j)$.

$$\theta_1{}^4 = A\ (1,6)\,(3,6)\,(5,6)\,(1,3)\,(1,5)\,(3,5)\,(2,4)\,(2,7)\,(4,7)$$
$$\theta_2{}^4 = A\ (3,6)\,(5,6)\,(3,5)\,(1,2)\,(1,4)\,(2,4)\,(3,7)\,(5,7)$$
$$\theta_3{}^4 = A\ (3,6)\,(4,6)\,(3,4)\,(1,2)\,(1,5)\,(2,5)\,(1,7)\,(2,7)\,(5,7)$$
$$\theta_4{}^4 = A\ (2,6)\,(3,6)\,(5,6)\,(2,3)\,(2,5)\,(3,5)\,(1,4)\,(1,7)\,(4,7)$$
$$\theta_5{}^4 = A\ (4,6)\,(5,6)\,(4,5)\,(1,2)\,(1,3)\,(2,3)\,(1,7)\,(2,7)\,(3,7)$$
$$\theta_6{}^4 = A\ (1,6)\,(2,6)\,(3,4)\,(3,5)\,(4,5)\,(1,2)\,(1,7)\,(2,7)$$
$$\theta_7{}^4 = A\ (2,6)\,(3,6)\,(4,6)\,(1,5)\,(2,3)\,(2,4)\,(3,4)\,(1,7)\,(5,7)$$
$$\theta_8{}^4 = A\ (2,6)\,(3,6)\,(2,3)\,(1,4)\,(1,5)\,(4,5)\,(1,7)\,(4,7)\,(5,7)$$
$$\theta_9{}^4 = A\ (1,6)\,(3,6)\,(1,3)\,(2,4)\,(2,5)\,(4,5)\,(1,7)\,(3,7)$$
$$\theta_{10}{}^4 = A\ (3,6)\,(5,6)\,(3,5)\,(1,2)\,(1,4)\,(2,4)\,(1,7)\,(2,7)\,(4,7)$$
$$\theta_{11}{}^4 = A\ (3,6)\,(4,6)\,(5,6)\,(3,4)\,(3,5)\,(4,5)\,(1,2)\,(1,7)\,(2,7)$$
$$\theta_{13}{}^4 = A\ (2,6)\,(4,6)\,(5,6)\,(1,3)\,(2,4)\,(2,5)\,(4,5)\,(1,7)\,(3,7)$$
$$\theta_{14}{}^4 = A\ (2,6)\,(5,6)\,(2,5)\,(1,3)\,(1,4)\,(3,4)\,(1,7)\,(3,7)\,(4,7)$$
$$\theta_{15}{}^4 = A\ (1,6)\,(5,6)\,(1,5)\,(2,3)\,(2,4)\,(3,4)\,(1,7)\,(5,7)$$
$$\theta_{16}{}^4 = A\ (1,6)\,(2,3)\,(2,4)\,(2,5)\,(3,4)\,(3,5)\,(4,5)\,(1,7)$$
$$\theta_{17}{}^4 = A\ (1,6)\,(4,6)\,(2,3)\,(2,5)\,(3,5)\,(1,4)\,(1,7)\,(4,7)$$
$$\theta_{18}{}^4 = A\ (2,6)\,(4,6)\,(1,3)\,(1,5)\,(3,5)\,(2,4)\,(1,7)\,(3,7)\,(5,7)$$
$$\theta_{19}{}^4 = A\ (3,6)\,(4,6)\,(1,2)\,(1,5)\,(2,5)\,(3,4)\,(3,7)\,(4,7)$$
$$\theta_{20}{}^4 = A\ (2,6)\,(1,3)\,(1,4)\,(1,5)\,(3,4)\,(3,5)\,(4,5)\,(2,7)$$
$$\theta_{21}{}^4 = A\ (1,6)\,(4,6)\,(5,6)\,(1,4)\,(1,5)\,(4,5)\,(2,3)\,(2,7)\,(3,7)$$
$$\theta_{22}{}^4 = A\ (1,6)\,(3,6)\,(4,6)\,(1,3)\,(1,4)\,(3,4)\,(2,5)\,(2,7)\,(5,7)$$
$$\theta_{23}{}^4 = A\ (1,6)\,(2,6)\,(3,4)\,(3,5)\,(4,5)\,(1,2)\,(3,7)\,(4,7)\,(5,7)$$
$$\theta_{24}{}^4 = A\ (4,6)\,(5,6)\,(1,2)\,(1,3)\,(2,3)\,(4,5)\,(4,7)\,(5,7)$$
$$\theta_{25}{}^4 = A\ (3,6)\,(1,2)\,(1,4)\,(1,5)\,(2,4)\,(2,5)\,(4,5)\,(3,7)$$
$$\theta_{26}{}^4 = A\ (2,6)\,(4,6)\,(1,3)\,(1,5)\,(3,5)\,(2,4)\,(2,7)\,(4,7)$$
$$\theta_{27}{}^4 = A\ (1,6)\,(5,6)\,(1,5)\,(2,3)\,(2,4)\,(3,4)\,(2,7)\,(3,7)\,(4,7)$$
$$\theta_{28}{}^4 = A\ (1,6)\,(3,6)\,(1,3)\,(2,4)\,(2,5)\,(4,5)\,(2,7)\,(4,7)\,(5,7)$$
$$\theta_{29}{}^4 = A\ (1,6)\,(2,6)\,(4,6)\,(3,5)\,(1,2)\,(1,4)\,(2,4)\,(3,7)\,(5,7)$$
$$\theta_{30}{}^4 = A\ (5,6)\,(1,2)\,(1,3)\,(1,4)\,(2,3)\,(2,4)\,(3,4)\,(5,7)$$
$$\theta_{31}{}^4 = A\ (1,6)\,(2,6)\,(3,6)\,(1,2)\,(1,3)\,(2,3)\,(4,5)\,(4,7)\,(5,7)$$

$$
\begin{aligned}
{\theta_{32}}^4 &= A\ (1,6)\,(4,6)\,(2,3)\,(2,5)\,(3,5)\,(1,4)\,(2,7)\,(3,7)\,(5,7)\\
{\theta_{33}}^4 &= A\ (2,6)\,(5,6)\,(1,3)\,(1,4)\,(3,4)\,(2,5)\,(2,7)\,(5,7)\\
{\theta_{34}}^4 &= A\ (2,6)\,(3,6)\,(1,4)\,(1,5)\,(4,5)\,(2,3)\,(2,7)\,(3,7)\\
{\theta_{35}}^4 &= A\ (4,6)\,(1,2)\,(1,3)\,(1,5)\,(2,3)\,(2,5)\,(3,5)\,(4,7)\\
{\theta_{36}}^4 &= A\ (1,6)\,(2,6)\,(5,6)\,(1,2)\,(1,5)\,(2,5)\,(3,4)\,(3,7)\,(4,7)
\end{aligned}
$$

By using the set of equations given above we have several choices for $a_1, \cdots, a_5$ in terms of theta constants.

Branch Points	Possible Ratios		
a_1^2	$\left(\frac{\theta_{36}^2\theta_{22}^2}{\theta_{33}^2\theta_{19}^2}\right)^2$	$\left(\frac{\theta_{31}^2\theta_{21}^2}{\theta_{34}^2\theta_{24}^2}\right)^2$	$\left(\frac{\theta_{29}^2\theta_{1}^2}{\theta_{26}^2\theta_{2}^2}\right)^2$
a_2^2	$\left(\frac{\theta_{4}^2\theta_{29}^2}{\theta_{2}^2\theta_{17}^2}\right)^2$	$\left(\frac{\theta_{36}^2\theta_{7}^2}{\theta_{15}^2\theta_{19}^2}\right)^2$	$\left(\frac{\theta_{31}^2\theta_{13}^2}{\theta_{9}^2\theta_{24}^2}\right)^2$
a_3^2	$\left(\frac{\theta_{4}^2\theta_{22}^2}{\theta_{33}^2\theta_{17}^2}\right)^2$	$\left(\frac{\theta_{11}^2\theta_{31}^2}{\theta_{24}^2\theta_{6}^2}\right)^2$	$\left(\frac{\theta_{7}^2\theta_{1}^2}{\theta_{26}^2\theta_{15}^2}\right)^2$
a_4^2	$\left(\frac{\theta_{11}^2\theta_{29}^2}{\theta_{2}^2\theta_{6}^2}\right)^2$	$\left(\frac{\theta_{21}^2\theta_{7}^2}{\theta_{15}^2\theta_{34}^2}\right)^2$	$\left(\frac{\theta_{22}^2\theta_{13}^2}{\theta_{9}^2\theta_{33}^2}\right)^2$
a_5^2	$\left(\frac{\theta_{4}^2\theta_{21}^2}{\theta_{34}^2\theta_{17}^2}\right)^2$	$\left(\frac{\theta_{11}^2\theta_{36}^2}{\theta_{19}^2\theta_{6}^2}\right)^2$	$\left(\frac{\theta_{13}^2\theta_{1}^2}{\theta_{26}^2\theta_{9}^2}\right)^2$

Let's select the following choices for $a_1, \cdots, a_5$.

$$
a_1 = \frac{\theta_{31}^2\theta_{21}^2}{\theta_{34}^2\theta_{24}^2},\ a_2 = \frac{\theta_{31}^2\theta_{13}^2}{\theta_{9}^2\theta_{24}^2},\ a_3 = \frac{\theta_{11}^2\theta_{31}^2}{\theta_{24}^2\theta_{6}^2},\ a_4 = \frac{\theta_{21}^2\theta_{7}^2}{\theta_{15}^2\theta_{34}^2},\quad a_5 = \frac{\theta_{13}^2\theta_{1}^2}{\theta_{26}^2\theta_{9}^2}.
$$

This completes the proof. □

Remark 18. *Unlike the genus 2 case, here only θ_1, θ_6, θ_7, θ_{11}, θ_{15}, θ_{24}, θ_{31} are from one of the Göpel groups.*

4.1.1. *Genus 3 non-hyperelliptic cyclic curves.* Using the Thomae's like formula for cyclic curves, for each cyclic curve $y^n = f(x)$ one can express the roots of $f(x)$ in terms of ratios of theta functions as in the hyperelliptic case. In this section we study such curves for $g = 3$. We only consider the families of curves with positive dimension since the curves which belong to 0-dimensional families are well known. The proof of the following lemma can be found in [12].

Lemma 19. *Let f be a meromorphic function on $\mathcal{X}$, and let*

$$
(f) = \sum_{i=1}^{m} b_i - \sum_{i=1}^{m} c_i
$$

be the divisor defined by f. Let's take paths from P_0 (initial point) to b_i and P_0 to c_i so that $\sum_{i=1}^m \int_{P_0}^{b_i} \omega = \sum_{i=1}^m \int_{P_0}^{c_i} \omega$.

For an effective divisor $P_1 + \cdots + P_g$ we have

$$
f(P_1)\cdots f(P_g) = \frac{1}{E} \prod_{k=1}^{m} \frac{\theta(\sum_i \int_{P_0}^{P_i} \omega - \int_{P_0}^{b_k} \omega - \triangle, \tau)}{\theta(\sum_i \int_{P_0}^{P_i} \omega - \int_{P_0}^{c_k} \omega - \triangle, \tau)} \tag{15}
$$

where E is a constant independent of $P_1, \ldots, P_g$, the integrals from P_0 to P_i take the same paths both in the numerator and in the denominator, $\triangle$ denotes the Riemann's constant and $\int_{P_0}^{P_i} \omega = \left(\int_{P_0}^{P_i} \omega_1, \ldots, \int_{P_0}^{P_i} \omega_g\right)^t$.

Notice that the definition of thetanulls is different in this part from the definitions of the hyperelliptic case. We define the following three theta constants.

$$\theta_1 = \theta \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \quad \theta_2 = \theta \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad \theta_3 = \theta \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \end{bmatrix}$$

Next we consider the cases 16, 18, 19 from Table 4.

Case 18: If the automorphism group is C_3 then the equation of $\mathcal{X}$ is given by

$$y^3 = x(x-1)(x-s)(x-t).$$

Let Q_i where $i = 1..5$ be ramifying points in the fiber of $0, 1, s, t, \infty$ respectively. Consider the meromorphic function $f = x$ on $\mathcal{X}$ of order 3. Then we have $(f) = 3Q_1 - 3Q_5$. By applying the Lemma 19 with $P_0 = Q_5$ and an effective divisor $2Q_2 + Q_3$ we have the following.

$$Es = \prod_{k=1}^{3} \frac{\theta(2\int_{Q_5}^{Q_2}\omega + \int_{Q_5}^{Q_3}\omega - \int_{Q_5}^{b_k}\omega - \triangle, \tau)}{\theta(2\int_{Q_5}^{Q_2}\omega + \int_{Q_5}^{Q_3}\omega - \triangle, \tau)} \tag{16}$$

Again apply the Lemma 19 with an effective divisor Q_2+2Q_3 we have the following.

$$Es^2 = \prod_{k=1}^{3} \frac{\theta(\int_{Q_5}^{Q_2}\omega + 2\int_{Q_5}^{Q_3}\omega - \int_{Q_5}^{b_k}\omega - \triangle, \tau)}{\theta(\int_{Q_5}^{Q_2}\omega + 2\int_{Q_5}^{Q_3}\omega - \triangle, \tau)} \tag{17}$$

By dividing Eq. (17) by Eq. (16) we have,

$$\begin{aligned} s = &\prod_{k=1}^{3} \frac{\theta(\int_{Q_5}^{Q_2}\omega + 2\int_{Q_5}^{Q_3}\omega - \int_{Q_5}^{b_k}\omega - \triangle, \tau)}{\theta(\int_{Q_5}^{Q_2}\omega + 2\int_{Q_5}^{Q_3}\omega - \triangle, \tau)} \\ &\times \prod_{k=1}^{3} \frac{\theta(2\int_{Q_5}^{Q_2}\omega + \int_{Q_5}^{Q_3}\omega - \triangle, \tau)}{\theta(2\int_{Q_5}^{Q_2}\omega + \int_{Q_5}^{Q_3}\omega - \int_{Q_5}^{b_k}\omega - \triangle, \tau)} \end{aligned} \tag{18}$$

By a similar argument we have

$$\begin{aligned} t = &\prod_{k=1}^{3} \frac{\theta(\int_{Q_5}^{Q_2}\omega + 2\int_{Q_5}^{Q_4}\omega - \int_{Q_5}^{b_k}\omega - \triangle, \tau)}{\theta(\int_{Q_5}^{Q_2}\omega + 2\int_{Q_5}^{Q_4}\omega - \triangle, \tau)} \\ &\times \prod_{k=1}^{3} \frac{\theta(2\int_{Q_5}^{Q_2}\omega + \int_{Q_5}^{Q_4}\omega - \triangle, \tau)}{\theta(2\int_{Q_5}^{Q_2}\omega + \int_{Q_5}^{Q_4}\omega - \int_{Q_5}^{b_k}\omega - \triangle, \tau)} \end{aligned} \tag{19}$$

Computing the right hand side of Eq. (18) and Eq. (19) was the one of the main points of [11]. Finally, we have

$$s = \frac{\theta_2^3}{\theta_1^3}, \;\; and \;\; r = \frac{\theta_3^3}{\theta_1^3}.$$

Case 19: If the group is C_6 then the equation is $y^3 = x(x-1)(x-s)(x-t)$ with $s = 1-t$. By using results from case 18, we have

$$\theta_2^3 = \theta_1^3 - \theta_3^3.$$

Case 16: In this case the equation of $\mathcal{X}$ is given by

$$y^4 = x(x-1)(x-t).$$

This curve has 4 ramifying points Q_i where $i = 1..4$ in the fiber of $0, 1, t, \infty$ respectively. Consider the meromorphic function $f = x$ on $\mathcal{X}$ of order 4. Then we have $(f) = 4Q_1 - 4Q_4$. By applying the Lemma 19 with $P_0 = Q_4$ and an effective divisor $2Q_2 + Q_3$ we have the following.

$$Et = \prod_{k=1}^{4} \frac{\theta(2\int_{Q_4}^{Q_2}\omega + \int_{Q_4}^{Q_3}\omega - \int_{Q_4}^{b_k}\omega - \triangle, \tau)}{\theta(2\int_{Q_4}^{Q_2}\omega + \int_{Q_4}^{Q_3}\omega - \triangle, \tau)} \tag{20}$$

Again apply the Lemma 19 with an effective divisor Q_2+2Q_3 we have the following.

$$Et^2 = \prod_{k=1}^{4} \frac{\theta(\int_{Q_4}^{Q_2}\omega + 2\int_{Q_4}^{Q_3}\omega - \int_{Q_4}^{b_k}\omega - \triangle, \tau)}{\theta(\int_{Q_4}^{Q_2}\omega + 2\int_{Q_4}^{Q_3}\omega - \triangle, \tau)} \tag{21}$$

We have the following by dividing Eq. (21) by Eq. (20)

$$\begin{aligned} t = &\prod_{k=1}^{4} \frac{\theta(\int_{Q_4}^{Q_2}\omega + 2\int_{Q_4}^{Q_3}\omega - \int_{Q_4}^{b_k}\omega - \triangle, \tau)}{\theta(\int_{Q_4}^{Q_2}\omega + 2\int_{Q_4}^{Q_3}\omega - \triangle, \tau)} \\ &\times \prod_{k=1}^{4} \frac{\theta(2\int_{Q_4}^{Q_2}\omega + \int_{Q_4}^{Q_3}\omega - \triangle, \tau)}{\theta(2\int_{Q_4}^{Q_2}\omega + \int_{Q_4}^{Q_3}\omega - \int_{Q_4}^{b_k}\omega - \triangle, \tau)} \end{aligned} \tag{22}$$

In order to compute the explicit formula for t one has to find the integrals on the right hand side. Such computations are long and tedious and we intend to include them in further work.

Remark 20. *In the case 16) of Table 4, the parameter t is given by*

$$\theta[e]^4 = A(t-1)^4 t^2,$$

where $[e]$ is the theta characteristics corresponding to the partition $(\{1\}, \{2\}, \{3\}, \{4\})$ and A is a constant; see [8] for details. However, this is not satisfactory since we would like t as a rational function in terms of theta. The methods in [8] do not lead to another relation among t and the thetanulls since the only partition we could take is the above.

Summarizing all of the above we have:

Lemma 21. *Let $\mathcal{X}$ be a non-hyperelliptic genus 3 curve. The following are true:*

i): *If $Aut(\mathcal{X}) \cong C_3$, then $\mathcal{X}$ is isomorphic to a curve with equation*

$$y^3 = x(x-1)\left(x - \frac{\theta_2^3}{\theta_1^3}\right)\left(x - \frac{\theta_3^3}{\theta_1^3}\right).$$

ii): *If $Aut(\mathcal{X}) \cong C_6$, then $\mathcal{X}$ is isomorphic to a curve with equation*

$$y^3 = x(x-1)\left(x - \frac{\theta_2^3}{\theta_1^3}\right)\left(x - \frac{\theta_3^3}{\theta_1^3}\right) \text{ with } \theta_2^3 = \theta_1^3 - \theta_3^3.$$

iii): *If $Aut(\mathcal{X})$ is isomorphic to the group with GAP identity $(16, 13)$, then $\mathcal{X}$ is isomorphic to a curve with equation*

$$y^4 = x(x-1)(x-t) \text{ with}$$

where t is given by Eq. (22).

It seems possible to generalize such techniques of computing the branch points in terms of the theta functions for any cyclic cover of the projective line. We intend to pursue the ideas of these papers in further work.

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