

ON SOME ORDER 6 NON-SYMPLECTIC AUTOMORPHISMS OF ELLIPTIC K3 SURFACES.

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ABSTRACT. We classify non-symplectic automorphisms of order 6 on elliptic K3 surfaces which commute with a given elliptic fibration. We show how their study can be reduced to the study of non-symplectic automorphisms of order 3 and to a local analysis of the fibers. In particular, we determine the possible fixed loci and give their location on the singular fibers. When the Picard lattice is fixed, we show that K3 surfaces come in mirror pairs.

1. INTRODUCTION

An automorphism of a K3 surface is called non-symplectic when the induced action on the holomorphic 2-form is non-trivial. The study of non-symplectic automorphisms was pioneered by Nikulin [Nik81] who analyzed the case of involutions. Since then, these automorphisms have been extensively studied by several authors. Let us mention Vorontsov [Vor83], Kondō [Kon86, Kon92], Xiao Gang [Xia96], Machida and Oguiso [MO98], Oguiso and Zhang [OZ98, OZ00], Zhang [Zha07], Artebani and Sarti [AS08], and Artebani, Sarti and Taki [AST11]. From these works, we now know that if a K3 surface admits a non-symplectic automorphism, then the surface is algebraic and the Euler totient function evaluated at the order of the automorphism is at most 66. Moreover, non-symplectic automorphisms of prime order have been classified, a synthetic classification can be found in [AST11], and some authors have started to investigate the simultaneous existence of symplectic and non-symplectic automorphisms [Fra11].

One of the reasons behind the interest in non-symplectic involutions is the mirror

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construction of Borcea [Bor97] and Voisin [Voi93]. They construct Calabi-Yau and an explicit mirror map using, as building blocks K3 surfaces with non-symplectic involutions, and elliptic curves. This construction can be extended to K3 surfaces with non-symplectic automorphisms of order 3, 4 and 6 [Dil06].

In this paper, using the classification of non-symplectic automorphisms of order 3 [AS08], we study order 6 automorphisms of elliptic K3 surfaces commuting with the fibration by performing a combinatorial analysis of the action on the fixed locus.

2. PLAN

In Section 3, we define primitive non-symplectic automorphisms and fix the notation for the rest of the paper. In Section 4, we give the final classification of all possible fixed loci. In Section 5, we show that the fixed locus of a primitive non-symplectic automorphism of order 6 consists in a disjoint union of points, rational curves, and possibly one genus one curve. Our cases fall thus in two distinct situations which are analyzed in Sections 6 and 7. In the last Section, 8, we focus on the special case where ζ fixes the Picard lattice.

3. NOTATION

Let X be a smooth projective K3 surface and ζ an automorphism of X . The induced action of ζ on $H(X, \Omega^2) \simeq \mathbb{C}$ gives rise to a character χ . An automorphism is called *symplectic*, if it lies in the kernel of χ , and *non-symplectic* otherwise. If the order of ζ and $\chi(\zeta)$ agree, then ζ is called *primitive*. In the rest of the article, ζ will be a primitive non-symplectic automorphism of order 6 acting on X .

As suggested by Cartan [Car57], given a fixed point P of ζ , we can linearize the action around it. Since ζ is of order 6 and primitive, the linearized action can be written as

$$\begin{pmatrix} \xi_6^k & 0 \\ 0 & \xi_6^{k'} \end{pmatrix}$$

where $(k, k') \in \{(0, 1); (2, 5); (3, 4)\}$ and ξ_6 is a primitive 6th root of unity. While the first case corresponds to P lying on a fixed smooth curve, the last two options correspond to P being isolated. We will use the standard notation and say that P is of type $\frac{1}{6}(k, k')$. Since ζ is primitive, its iterates will also be non-symplectic. We will denote their fixed locus by $X^{[i]} = \{x \in X \text{ s.t. } \zeta^i x = x\}$. The components of the $X^{[i]}$ will be described by the following variables:

- $p_{\frac{1}{n}(k, k')}$: number of isolated fixed points of type $\frac{1}{n}(k, k')$ in $X^{[\frac{6}{n}]}$, for $n \in \{6, 3, 2\}$.
- $l^{[i]}$: number of rational curves in $X^{[i]}$.
- $g^{[i]}$: maximal genus among the curves in $X^{[i]}$.
- $g_M = \max\{1, g^{[1]}\}$.

When referring to [AS08], we will use their notation, namely:

- g : highest genus of the curves in $X^{[2]}$.
- n : number of fixed points in $X^{[2]}$ (all are of type $\frac{1}{3}(2, 2)$).
- k : total number of curves in $X^{[2]}$.

4. RESULTS

Our first result is a global description of the fixed locus of ζ .

Theorem 4.1. *The fixed locus $X^{[1]}$ consists of one of the two following collections:*

1. *a smooth genus 1 curve and three isolated fixed points of type $\frac{1}{6}(2, 5)$.*
2. *a disjoint union of smooth rational curves and points, $C_1 \sqcup \dots \sqcup C_l \sqcup P_1 \sqcup \dots \sqcup P_{p_{\frac{1}{6}(3,4)} + p_{\frac{1}{6}(2,5)}}$, satisfying*

$$(4.1) \quad p_{\frac{1}{6}(3,4)} + 2p_{\frac{1}{6}(2,5)} - 6l^{[1]} = 6.$$

The proof of this Theorem follows from Section 5.

From our analysis in Sections 7 and 6 we obtain the following:

Classification 4.2. *Let $X \rightarrow \mathbb{P}^1$ be an elliptic K3 surface and ζ a primitive non-symplectic automorphism of order 6 preserving the elliptic fibration. The fixed locus of ζ is one of the configurations listed in Table 1 or consists of the disjoint union of a genus 1 curve and three isolated points.*

Reading Table 1: First note that since ζ preserves the fibration, we have an induced action, ψ , on the basis. The order of ψ is either one or two. In the Table, each row begins by a description of $X^{[2]}$, the fixed locus of ζ^2 . After that comes a list of the singular fibers of the fibration (x is the number of fibers of type X). Then, we give a description of $X^{[1]}$ when ψ is the identity. Finally, the last two groups refer to the case where ψ is an involution; we list the fibers above the two fixed points, and the components of $X^{[1]}$.

5. STUDY OF THE FIXED LOCUS

Lemma 5.1. *The fixed locus of ζ consists of a disjoint union of smooth curves and points*

$$X^{[1]} = C_0 \sqcup \dots \sqcup C_m \sqcup P_1 \sqcup \dots \sqcup P_{p_{\frac{1}{6}(3,4)} + p_{\frac{1}{6}(2,5)}}$$

with $g(C_0) \geq 0 = g(C_1) = \dots = g(C_m)$.

Proof. The first part of the statement follows from the Hodge Index Theorem. The argument is analogue to those found in [Nik81, Voi93, Dil06, AS08]:

A disjoint union of smooth curves on a K3 surface can have at most one element with strictly positive self-intersection. By adjunction, that is a curve of genus at least 2.

If a curve has self-intersection 0, then it is an elliptic curve and induces an elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$. Since the action is non-symplectic, it descends non-trivially to the base and fixes two points. The fixed locus of ζ is thus a component of the fibers above these two points. One of the fibers is the original fixed curve. The remaining curves of the fixed locus are either a smooth elliptic curve or a disjoint union of rational components of one of Kodaira’s singular fibers. So either the fixed locus is as the one described in the statement, or it consists exactly in the disjoint union of two genus 1 curves. However, if $X^{[1]}$ were to contain two genus 1 curves, then so would $X^{[2]}$ and this option was ruled out in [AS08]. \square

Lemma 5.2. *The components of $X^{[1]}$ satisfy*

$$(5.1) \quad p_{\frac{1}{6}(3,4)} + 2p_{\frac{1}{6}(2,5)} - 6l^{[1]} + 6g_M = 12.$$

TABLE 1. Fixed locus when X is ζ -elliptic.

#	g	n	k	ii	iv	iv^*	iv^*	$p_{(3,4)}$	$p_{(2,5)}$	$l^{[1]} - 1$	F_0	F_∞	$p_{(3,4)}$	$p_{(2,5)}$	$l^{[1]}$
1	5	0	2	12	0	0	0	12	0	0	I_0	I_0	6	0	0
2	4	1	2	10	1	0	0	10	1	0	I_0	IV	4	1	0
3	3	2	2	8	2	0	0	8	2	0	I_0	I_0	6	0	0
4	2	3	2	6	3	0	0	6	3	0	I_0	IV	4	1	0
5	3	3	3	8	3	0	0	10	1	0	I_0	IV^*	6	3	1
6	1	4	2	4	4	0	0	4	4	0	I_0	I_0	6	0	0
7	2	4	3	4	4	0	0	8	2	0	IV	IV^*	4	4	1
8	3	4	4	6	1	0	0	10	4	1	I_0	IV	4	1	0
9	0	5	2	7	0	0	0	2	5	0	I_0	IV	4	1	0
10	1	5	3	4	5	0	0	2	3	0	I_0	IV^*	6	3	1
11	2	5	4	5	2	0	0	8	5	1	IV	IV^*	4	4	1
12	0	6	3	2	3	0	0	4	4	0	IV	IV^*	4	4	1
13	1	6	4	3	2	1	0	6	6	1	IV	IV^*	4	4	1
14	0	7	4	1	3	1	0	4	7	1	I_0	I_0	6	0	0
15	1	7	5	3	0	1	1	8	5	1	I_0	I_0	6	0	0
16	0	8	5	1	1	1	1	6	6	1	I_0	I_0	4	1	0
17	1	8	6	2	0	2	0	8	8	2	I_0	I_0	6	0	0
18	0	9	6	0	1	2	0	6	9	2	I_0	I_0	4	1	0
3'	3	2	2	8	2	0	0	8	2	0	IV	IV	2	2	0
6'	1	4	2	4	4	0	0	4	4	0	IV	IV	2	2	0

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Proof. The formula is simply the Lefschetz holomorphic formula, see [TT75], applied to ζ . \square

We will use the classification, determined by Artebani and Sarti [AS08], of non-symplectic automorphism of order 3 to find information on $X^{[2]}$, which in turn will yield us data on the nature of $X^{[1]}$. We first recapitulate the results of [AS08], which we will use later, and then relate the fixed loci $X^{[1]}$ and $X^{[2]}$.

Theorem 5.3. [AS08, Proposition 4.2] *Let σ be a non-symplectic automorphism of order 3 acting on a K3 surface X . If the fixed locus of σ contains two or more curves, then X is isomorphic to an elliptic K3 surface whose Weierstrass equation is*

$$y^2 = x^3 + p_{12}(t)$$

and on which σ acts as $(x, y, t) \mapsto (\zeta_3^2 x, y, t)$.

Proposition 5.4. [AS08, Corollary 4.3] *Let σ and X be as in the statement of Theorem 5.3. If $X^{[1]}$ contains a curve C of positive genus, then C is a double section of the Weierstrass fibration, i.e. C is hyperelliptic.*

Lemma 5.5. *If $P \in X^{[1]}$ is of type $\frac{1}{6}(2, 5)$, then it is also an isolated point in $X^{[2]}$. If $P \in X^{[1]}$ is of type $\frac{1}{6}(3, 4)$, then it lies on a smooth curve in $X^{[2]}$. Moreover, one has the following inequalities $p_{\frac{1}{3}(2,2)} \geq p_{\frac{1}{6}(2,5)}$ and $l^{[2]} \geq l^{[1]}$.*

Proof. The first two statements are obvious after one takes the square of the matrix giving the localized action of ζ at P . The inequalities ensue. \square

Corollary 5.6. *If $X^{[1]}$ contains at least two distinct curves or a curve and an isolated point of type $\frac{1}{6}(3, 4)$, or more generally, if $X^{[2]}$ contains at least two distinct curves, then X is isomorphic to an elliptic K3 surface whose Weierstrass equation is*

$$y^2 = x^3 + p_{12}(t).$$

Proof. Lemma 5.5 implies that $X^{[2]}$ contains at least two distinct curves. The first part of the statement follows thus directly from Theorem 5.3. \square

In the rest of the paper, we will focus on elliptic K3 surfaces. The following two Lemmas show that this is a somewhat mild restriction.

Lemma 5.7. *Let X be a K3 surface and τ a non-symplectic automorphism of order 3 which preserves the fibration. Then there exists a primitive non-symplectic automorphism ζ of X such that $\tau = \zeta^2$ and X is ζ -elliptic.*

Proof. We know from 5.3 that X is of the form $y^2 = x^3 + p_{12}(t)$ and τ acts as $(x, y, t) \mapsto (\xi_3 x, y, t)$. It is easy to see that $\sigma : (x, y, t) \mapsto (\xi_3 x, -y, t)$ acts on X and has the required properties. \square

Lemma 5.8. *If the fixed locus of a non-symplectic automorphism of order 6 contains a rational curve then X is elliptic.*

Proof. When the fixed locus contains at least one curve, formula 5.1 reduces to $p_{\frac{1}{6}(3,4)} + 2p_{\frac{1}{6}(2,5)} - 6l^{[1]} = 6$. If $p_{\frac{1}{6}(3,4)}$ is strictly positive, then Corollary 5.6 implies that X is elliptic. Otherwise, $p_{(2,5)} \geq 6$ and thus n is an odd number greater than or equal to 6. From [AS08, Table 2] one can see that all cases where $X^{[2]}$ contains a rational curve and where n is an odd number larger than 6 are elliptic. \square

While Lemma 5.7 shows that any automorphism of order 3 preserving the fibration factors through an automorphism of order 6, this automorphism does not have to be unique. It is possible that a generic automorphism of order 6 when applied twice gives an automorphism of order 3 commuting with the fibration. From now on we will focus on automorphisms of order 6 that actually do preserve the elliptic fibration:

Definition 5.9. *If X, ζ are as in the statement of Corollary 5.6, we will say that X is ζ -elliptic if ζ preserves the elliptic fibration.*

In this situation, we have an induced action on the basis which we will denote by $\psi = \pi \circ \zeta \circ \pi^{(-1)}$. Indeed, the action of ζ preserves the fibration, i.e. $\pi \circ \zeta \circ \pi^{(-1)}$ is well defined, and ζ^2 acts as $(x, y, t) \mapsto (\zeta_3^2 x, y, t)$. In particular, this implies that the induced action on the base is at most of order 2 and if this induced action is trivial, then ζ restricts to an action of order 6 on each fiber.

An important property of automorphisms which preserve fibrations is given by the following statement:

Lemma 5.10. *If X is ζ -elliptic then $X^{[i]}$ does not contain curves of strictly positive genus.*

Proof. Assume that $X^{[1]}$ contains a curve C_0 which is not rational. Proposition 5.4 tells us that C_0 is a double section of the fibration and therefore, the action induced on the base is trivial. We are then in the situation where ζ induces an automorphism of order 6 on each fiber. Moreover, since ζ fixes at least two points per fiber, the points of intersection with C_0 , it ought to be the identity: a contradiction. \square

Remark. If X is ζ -elliptic, Lemma 5.10 tells us that the fixed locus of ζ contains no curves of positive genus. However, one could have an automorphism of order 6 commuting with an elliptic fibration and fixing a curve of genus one. Indeed, the proof of the Lemma does not exclude the fixed curve to be a fiber itself.

We can thus conclude

Conclusion 5.11. *The fixed locus of ζ consists either of a disjoint union of smooth rational curves and points or of a configuration containing possibly one elliptic curve. In Section 6.2, we show that the genus one situation is actually unique; all other cases, which correspond to X ζ -elliptic, are discussed in Section 7.*

6. THE CASE WHERE THE FIXED LOCUS CONTAINS A GENUS 1 CURVE

Lemma 6.1. *If $g(C_0) = 1$ then $p_{\frac{1}{6}(3,4)} = l^{[1]} = 0$, $p_{\frac{1}{6}(2,5)} = 3$ and $l^{[2]} = 0$, $p_{\frac{1}{3}(2,2)} = 3$.*

Proof. If $l^{[1]}$ or $p_{\frac{1}{6}(3,4)}$ were to be strictly positive, Corollary 5.6 would imply that X is ζ -elliptic contradicting Lemma 5.10. Formula 5.1 gives us the value of $p_{\frac{1}{6}(2,5)}$. Similarly, the case $l^{[2]} > 0$ is excluded as we would reach a similar contradiction. Finally, the value for $p_{\frac{1}{3}(2,2)} = 3$ can be found in [AS08, Table 1]. \square

A non-symplectic automorphism of order 6 which fixes a smooth elliptic curve fixes thus also three isolated points and nothing else. Actually,

Proposition 6.2. *If the fixed locus of a non-symplectic automorphism contains an elliptic curve C_0 , then the action is defined uniquely i.e., the fixed loci of ζ , ζ^2 and ζ^3 are determined uniquely.*

- *The fixed locus of ζ and ζ^2 are identical : $X^{[1]}$ and $X^{[2]}$ consist of C_0 and three isolated points – as described in the previous Lemma.*
- *The fixed locus of ζ^3 is a superset of the previous fixed loci : it consists of C_0 and a second smooth elliptic curve C_1 .*

Proof. Consider the elliptic fibration given by the linear system $|C_0|$. Since C_0 is in the fixed locus, the induced action on the base is of order 6, i.e. it is a cyclic action with two fixed points: the image of C_0 and some additional point Q . Since the Euler characteristic of a K3 surface is 24, the Euler characteristic of the fiber above Q is a multiple of 6. From Kodaira’s classification of the possible singular fibers, the fiber above Q is of the type I_{6N} or I_{6N}^* . However, as will follow from Section 7.1, only in the case I_0 does ζ not fix any rational curves. The fiber above Q is thus smooth and the fixed loci of ζ and its powers are readily found. \square

Remark. An example of a K3 with a primitive non-symplectic automorphism of order 6 fixing an elliptic curve is given by the surface $y^2 = x^3 + (t^6 - 1)^2$, where the action is $\zeta : (x, y, t) \mapsto (x, y, \xi_6 t)$. The volume form $\omega = \frac{dx \wedge dt}{dy}$ gets mapped to $\zeta^* \omega = \xi_6 \omega$.

7. ELLIPTIC CASE

In this Section we consider X to be ζ -elliptic. The induced automorphism, ψ , on \mathbb{P}^1 , is either trivial or an involution. The two cases are analyzed respectively in Sections 7.2 and 7.3. Our discussion begins in Section 7.1 where we analyze how ζ acts on the fibers of π .

7.1. Local analysis. Let X be a K3 surface. The *Gram graph* of X is the incidence graph of the effective smooth rational curves on X . E.g., when the Picard lattice of X is isomorphic to $U \oplus E_8^2$ of S_X , then the Gram graph is as in figure 1. Let D be an effective divisor on X and ζ an automorphism of X . We call D *stable* if $\zeta(D) = D$, and we say that D is *fixed* if $\zeta|_D = \text{id}$.

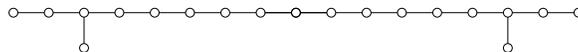


FIGURE 1. Gram graph of $U \oplus E_8^2$.

Lemma 7.1. *Consider a tree of rational curves on a surface X which are stable componentwise under the action of a primitive non-symplectic automorphism of order 6. Then, the points of intersection of the rational curves are fixed and the action at one fixed point determines the action on the whole tree.*

Proof. The key in this proof is to realize that the action of the automorphism on a given rational component and the action on a fixed point of this curve determine each other completely. Recall that an action of \mathbb{C} will be of the form $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$ under suitable coordinates. Now, λ is nothing but the eigenvalue associated to the fixed point of coordinate 0, or the inverse of the eigenvalue associated to the fixed

point at infinity. Conversely, if one knows one eigenvalue of the automorphism localized at a point, then one knows the full action at that point. First, the eigendirections correspond to the components of the tree passing through the point. Second, since the three types of points, $\frac{1}{6}(3, 4)$, $\frac{1}{6}(2, 5)$ and $\frac{1}{6}(1, 0)$, all have distinct eigenvalues it is clear to which eigenvalue corresponds each direction. \square

Remark. It follows from the proof of the previous lemma that if we look at the types of points of intersection on a chain of smooth rational curves, these will embed in the following periodic sequence:

$$\dots, \frac{1}{6}(2, 5), \frac{1}{6}(3, 4), \frac{1}{6}(3, 4), \frac{1}{6}(2, 5), \frac{1}{6}(1, 0), \frac{1}{6}(1, 0), \dots$$

Example 7.2. Consider a type IV^* configuration of rational curves which is stable under the action of ζ , a non-symplectic automorphism of order 6. Moreover, assume that it contains on one of the weight 1 curves, L , a point P of type $\frac{1}{6}(3, 4)$, such that the eigendirection corresponding to the eigenvalue -1 is transversal to the L . Using Lemma 7.1 we can determine the action on the entire configuration. This action is illustrated in Figure 2.

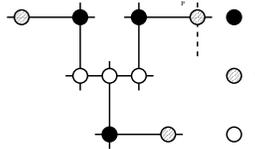


FIGURE 2. Study of the action on a type IV^* configuration.

From now on, we will focus only on fibers of type I_0 , II , II^* , IV and IV^* . We will denote by ii the number of type II fibers, ii^* the number of type II^* fibers, etc. We focus our attention on these fibers because of the following theorem:

Proposition 7.3. [AS08, Proposition 4.2]) Let X be ζ -elliptic, then the numbers (n, k) determine uniquely ii , ii^* , iv and iv^* . More precisely, π has

1. n type IV fibers if $k = 2$.
2. $n - 3$ type IV fibers and 1 type IV^* fiber if $k = 3$.
3. $n - 4$ type IV fibers and 1 type II^* fiber if $k = 4$.
4. $n - 7$ type IV fibers, 1 type IV^* fiber and 1 type II^* fiber if $k = 5$.
5. $n - 8$ type IV and 2 type II^* fibers if $k = 6$.

Lemma 7.4. Let X be ζ -elliptic, ψ trivial, and assume that X has a fiber of type II , IV , II^* or IV^* . When restricted to those fibers, ζ

1. fixes 1 point of type $\frac{1}{6}(3, 4)$, namely the cuspidal point of the fiber. (Fiber of type II)
2. fixes 3 points of type $\frac{1}{6}(3, 4)$, 4 points of type $\frac{1}{6}(2, 5)$ and 1 rational curve. (Fiber of type II^*)
3. fixes 1 point of type $\frac{1}{6}(2, 5)$, namely the common intersection point. (Fiber of type IV)
4. fixes 2 points of type $\frac{1}{6}(3, 4)$ and 1 point of type $\frac{1}{6}(2, 5)$. (Fiber of type IV^*)

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Moreover, π has also a section fixed by ζ and it is the only part of $X^{[1]}$ not completely included in the fibers.

Proof. Since ψ is trivial, the fibers are preserved by ζ . Thus either ζ fixes the zero section σ_0 , or there is another section σ_1 and ζ permutes the two. Assume σ_0 is not fixed. Pick a smooth fiber F of π . The automorphism ζ^2 is of order 3 on F and fixes the 2 points of intersection with the two sections σ_0 and σ_1 . Therefore, there is a third fixed point. Since ζ permutes the first two, it fixes the third one. Since ψ is trivial, this point is of type $\frac{1}{6}(1, 0)$ and there is a fixed section passing through that point.

Let us describe the action on the fibers explicitly.

1. (II) The point of intersection with the fixed section is not the node, as the section intersects the fiber with multiplicity 1, and is of type $\frac{1}{6}(0, 1)$. On the other hand, the other fixed point, which ought to be the node, is of type $\frac{1}{6}(3, 4)$.
2. (II*) Since the Gram graph of this fiber has no non-trivial $\mathbb{Z}/2\mathbb{Z}$ automorphism, the curve of weight 6 is fixed. The remaining fixed points can be found using Lemma 7.1.
3. (IV) The section of the Weierstrass fibration is fixed and intersects the fiber at the curve of weight 1. Using Lemma 7.1 we see that there is a unique possible action, namely the one permuting the two other branches.
4. (IV*) The action on the fiber follows from lemma 7.1 and is described in figure 3. The black dot corresponds to a point of type $\frac{1}{6}(2, 5)$ and the two white dots to points of type $\frac{1}{6}(3, 4)$.

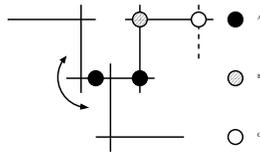


FIGURE 3. Action on a type IV^* fiber when ψ is trivial.

□

Lemma 7.5. Let X be ζ -elliptic and assume ψ is an involution. Let F be a fiber preserved by ζ , i.e. $\zeta(F) = F$. F is of type I_0 , IV or IV^* . Moreover, ζ

1. fixes 3 points of type $\frac{1}{6}(3, 4)$, when F is smooth.
2. fixes 1 point of type $\frac{1}{6}(3, 4)$ and 1 point of type $\frac{1}{6}(2, 5)$, when F is of type IV . This case is depicted in Example 7.2.
3. fixes 3 points of type $\frac{1}{6}(3, 4)$, 3 points of type $\frac{1}{6}(2, 5)$ and 1 rational curve, when F is of type IV^* .

Moreover, every component of $X^{[1]}$ lies in one of those fibers.

Proof. Without loss of generality, we can assume that ψ is of the form $[x_0 : x_1] \mapsto [-x_0 : x_1]$, or $t \mapsto -t$. Since, the Weierstrass equation $y^2 = x^3 + p_{12}(t)$ is invariant under ψ , this implies that the roots of p_{12} are double at 0 and ∞ . The fibers which correspond to double roots are those of type I_0 , IV and IV^* . Alternatively, one can perform a local analysis on the fibers, and see that these are the only

possibilities. This analysis will also give us the exact nature of the fixed locus on each fiber. Take a fixed point P at the intersection of the section of π and a fiber F . Since ψ is an involution, the eigenvalue corresponding to the direction of the section is -1 . Using Lemma 7.1 we can describe the local action in each case:

1. (I_0) There are 3 fixed points of type $\frac{1}{6}(3, 4)$.
2. (IV) There is 1 fixed point of type $\frac{1}{6}(3, 4)$, and one of type $\frac{1}{6}(2, 5)$.
3. (IV^*) There are 3 points of type $\frac{1}{6}(3, 4)$, 3 points of type $\frac{1}{6}(2, 5)$ and 1 rational curve.

□

7.2. Induced action on the base is trivial.

Lemma 7.6. *Let X be ζ -elliptic, ψ trivial. Then $X^{[2]}$ determines $X^{[1]}$. The possibilities are listed in Table 1.*

Proof. From Proposition 7.3, we know that (n, k) determines the types of fibers of π . Since, Lemma 7.4 tells us that the action of ζ on each fiber is unique, it follows that $X^{[1]}$ is completely determined by $X^{[2]}$. □

Remark. Unfortunately, the converse is not true: the simple combinatorial data describing $X^{[1]}$ does not determine uniquely X or $X[2]$. See examples 4 and 10 in Table 1.

Finally, the existence of all the examples in Table 1 follows from Lemma 5.7.

7.3. Induced action on the base is an involution.

Lemma 7.7. *The fixed locus $X^{[1]}$ is contained in 2 fibers of π .*

Proof. Since ψ is an involution, it has two fixed points on \mathbb{P}^1 , say 0 and ∞ . Since all the fibers not above these points are permuted, $X^{[1]}$ is a subset of the fibers F_0 and F_∞ ($F_i = \pi^{(-1)}(i)$). □

Lemma 7.8. *Let X be a K3 surface and τ a non-symplectic automorphism of order 3 which is τ -elliptic. Call π the associated fibration. Assume that the multiset X_π of singular fibers of π can be decomposed $F \sqcup M$ with F a multiset of cardinality 2 whose elements come from $\{I_0, IV, IV^*\}$ and where each element of M has even multiplicity. Then there exists a pair (X', τ') consisting of a K3 surface and a non-symplectic automorphism of order 3 such that X' is τ' -elliptic, $X_\pi = X'_{\pi'}$, and $X^\tau = X'^{\tau'}$. Moreover, τ' factors as $\tau' = \zeta^2$ where ζ is a primitive non-symplectic automorphism of order 6 commuting with π' .*

Proof. This follows from the local analysis in Lemma 7.5, or from the fact that the only singular fibers corresponding to double roots of $p_{12}(t)$ are those of I_0 , IV , and IV^* . □

Since the action on F_0 and F_∞ is determined by Lemma 7.5, we simply list all possibilities in Table 1.

8. FIXED PICARD LATTICE.

A special case of the above classification consists of analysing only those automorphisms which fix the Picard group. Although this can be recovered from the previous sections, we will try to analyse the case separately to make the analogy with automorphisms of order 2 and 3 as studied by [Nik81] and [AS08].

Recall that for a K3 surface, X , the cohomology $H^2(X, \mathbb{Z})$ is a unimodular lattice of signature $(3, 19)$ i.e., it is isomorphic to $U^3 \oplus E_8^2$. Also, it decomposes into the Picard lattice, S_X , and the transcendental lattice T_X :

$$H^2(X, \mathbb{Z}) \cong S_X \oplus T_X.$$

Given a lattice A , we will write A^\perp for its orthogonal complement and A^* for its dual $\text{Hom}(A, \mathbb{Z})$. We say that a lattice is *p-elementary* when $A^*/A = (\mathbb{Z}/p\mathbb{Z})^k$ for some $k \in \mathbb{N}$.

Lemma 8.1. *Let ζ be a primitive non-symplectic automorphism of order 6 of X which preserves the Picard lattice, then the Picard lattice $S_X = H^2(X, \mathbb{Z})^\zeta$ is a unimodular.*

Proof. Fix $p \in \{2, 3\}$, and let $\zeta_p = (\zeta^*)^{6/p}$. The quotients S_X^*/S_X and $(S_X^\perp)^*/S_X^\perp = T_X^*/T_X$ are isomorphic. Hence, $p = 1 + \zeta_p^* + \dots + (\zeta_p^*)^{p-1} = 0$ on T_X and $pT_X^* \subset T_X$. Since S_X is both 2 and 3 elementary, it is unimodular. \square

Corollary 8.2. *The Picard lattice S_X is isomorphic to U , $U \oplus E_8$ or $U \oplus E_8^2$.*

Proof. By the Hodge index theorem, S_X is of signature $(1, *)$. By adjunction, the lattice is even. Using the classification of even unimodular lattices, e.g. in [Ser70], we get the desired result. \square

Since in the three cases S_X decomposes as the direct sum of U with a negative definite lattice, it is easy to see that we fall everytime in the elliptic case. Moreover, using the Lefschetz topological formula or the fact that only the Picard lattice is fixed, one can see that there are no other irreducible fibers except for the given E_8 fibers generating part of the Picard lattice. Note that using lemma 7.1 one can see that in the case of $\text{rk } S_X > 2$ only the section and the rational lines of degree 3 are fixed.

Recall the following definition, due to Dolgachev [Dol96], of mirror pairs for K3 surfaces.

Definition 8.3. *The K3 surfaces (M, W) form a mirror pair whenever $S_M^\perp = S_W \oplus U$.*

When applied to the case of unimodular Picard lattices, we see that K3 surfaces form a pair when their Picard groups are respectively $U^i \oplus E_8^j$ and $U^{2-i} \oplus E_8^{2-j}$. I.e. the surfaces with Picard groups U and $U \oplus E_8^2$ are dual to one another while the surfaces with Picard group $U \oplus E_8$ are self-dual. This confirms the diagrams obtained for automorphisms of order 2 and 3 showing that mirror symmetry is a natural transformation preserving symmetries.

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