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AN UPPER BOUND FOR THE X-RANKS OF POINTS OF $\mathbb{P}^n$ IN POSITIVE CHARACTERISTIC

E. BALLICO

ABSTRACT. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ is the minimal cardinality of $S \subset X$ such that $P \in \langle S \rangle$. Here we study the pairs (X, P) such that $r_X(P) \geq n+2-m$, i.e. $r_X(P) = n+2-m$. These pairs exist only in positive characteristic, with X strange and P a strange point of X.

1. Introduction

Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-ranks are defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. For any integer $r > 0$ let $\sigma_r^0(X) \subseteq \mathbb{P}^n$ denote the the union all $(r-1)$-dimensional linear spaces spanned by r points of X. Let $\sigma_r(X)$ denote the closure of $\sigma_r^0(X)$ in $\mathbb{P}^n$ (sometimes called the $(r-1)$-secant variety of X). The border X-rank of a point $P \in \mathbb{P}^n$ is the minimal integer r such that $P \in \sigma_r(X)$. Each $\sigma_r(X)$ is irreducible. An easy estimate gives that either $\sigma_r(X) = \mathbb{P}^n$ or $\dim(\sigma_{r+1}(X)) > \dim(\sigma_r(X))$ ([1], 1.2). Hence $\sigma_x(X) = \mathbb{P}^n$, where $x := n - \dim(X)$. Moreover, either $\sigma_{r+1}(X) = \mathbb{P}^n$ or $\dim(\sigma_{r+1}(X)) \geq 2 + \dim(\sigma_r(X))$ ([1], Corollary 1.4). Even if $\sigma_x(X) = \mathbb{P}^n$ there may be points with X-rank $> x$. The main concern of this paper is to extend the basic estimate $r_X(P) \leq n - \dim(X)$ made in [15], Proposition 5.1, in characteristic zero to the case $p := \mathrm{char}(\mathbb{K}) > 0$, listing some exceptional pairs (X, P) for which $r_X(P) = n - \dim(X) + 1$ (e.g. take $(n, m, p) = (2, 1, 1)$, as X a smooth conic and as P its strange point ([10], Example IV.3.8.2); in this example every line through P intersects X in a unique point and hence we need 3 points of X to span a linear space containing P).

It is believed that the concept of X-rank may be useful for "real world applications". In the applications when X is a Veronese embedding of $\mathbb{P}^m$ the X-rank is also called the "structured rank" (this is related to the virtual array concept encountered in sensor array processing ([2], [8])). On this topic there was the 2008 AIM workshop Geometry and representation theory of tensors for computer science, statistics and other areas. In [15] a book in preparation is quoted ([14]). Up to now the applied part was toward engineering. All theory was done in characteristic

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zero. Our dream is to use these ideas together with specialists of computer algebra for real applications in coding theory. A preliminary step to fulfil this dream is to check the theory at least over an algebraically closed field with positive characteristic. Up to now the only general result on the X-rank (i.e. a result which does not use specific properties of very particular varieties X) is [15], Proposition 5.1. Hence its extension to positive characteristic seemed to be the first step needed to fulfil our dream. The aim of this paper is to prove that [15], Proposition 5.1, is not true in positive characteristic, but that it is "almost always true" and when it is not true it is "almost true" (it fails by +1). We also give a reasonable description of the projective varieties for which it is not true. The embedded variety $X \subseteq \mathbb{P}^n$ is said to be *strange* if there is $O \in \mathbb{P}^n$ such that $O \in T_QX$ (the embedded tangent space in $\mathbb{P}^n$) for all $Q \in X_{reg}$ (or, equivalently, for a general $Q \in X$) ([4]). If X is strange, a point as above is called a *strange point* of X. The set of all strange points of X is either empty or a linear subspace of dimension at most $\dim(X) - 1$ (unless $X = \mathbb{P}^n$). If $\mathrm{char}(\mathbb{K}) = 0$, then X is strange if and only if it is a cone and in this case the set of all strange points is its vertex (with the convention that a linear space is a cone with itself as its vertex). If X is strange with O as one of its strange points, but not a cone with vertex containing O, then $p := \mathrm{char}(\mathbb{K}) > 0$. If p is a large prime, then also $\deg(X)$ must be large (e.g. $\deg(X) \ge p(n-m)$) (see Proposition 3). We first prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $P \in \mathbb{P}^n$.*

(a) If P is not a strange point of X, then $r_X(P) \le n + 1 - m$.

(b) If P is a strange point of X, then $r_X(P) \le n + 2 - m$.

See Remark 4 for an example of an integral, non-degenerate and m-dimensional ($m \ge 2$) variety $X \subset \mathbb{P}^n$ with as strange points an $(m-1)$-dimensional linear space V and $r_X(P) = n - m + 2$ for all $P \in V \backslash N$, where N is a hyperplane of V and $N \subset X$.

The proof of Theorem 1 is very elementary. To prove Theorem 1 we just follow the proof of [15], Proposition 5.1 (the case $\mathrm{char}(\mathbb{K}) = 0$ of Theorem 1), analysing the only missing piece in positive characteristic (a use of Bertini's theorem). In the one-dimensional case we are able to improve Theorem 1. A non-degenerate curve $X \subset \mathbb{P}^n$ is said to be *very strange* if its general hyperplane section is not in linearly general position ([18]). A very strange curve is strange ([18], Lemma 1.1).

Definition 1. Let $X \subset \mathbb{P}^n$, $n \ge 2$, be a non-degenerate strange curve and let O be its strange point. Let $\ell_O : \mathbb{P}^n \backslash \{O\} \to \mathbb{P}^{n-1}$ be the linear projection from O and $T \subset \mathbb{P}^{n-1}$ the closure of $\ell_O(Y \backslash \{O\})$. Thus T is non-degenerate and

$$\deg(X) = p^e s \cdot \deg(T) + \mu, \tag{1}$$

where μ is the multiplicity of X at O, while s and p^e are the separable and the inseparable degree of $\ell_O|X$, respectively ([4], Theorem 2.3). Now assume $n \ge 3$, $\mu = 0$ (i.e. $O \notin X$) and $s = 1$. We say that X is *flat* or *flat with respect to its strange point* O or a *flat strange curve* if for any $S \subset X$ such that $\sharp(S) \le n$ we have $\dim(\langle S \rangle) = \dim(\langle \ell_O(S) \rangle)$.

The proofs that $e > 0$ in the set-up of Definition 1 and that (1) holds are given in [4], §2 (see [4], eq. (2.1.1) and Theorem 2.3); the integer p^e is shown to be equal to the intersection multiplicity of T_QX with X at Q, where Q is a general point of X

(the so-called Generic Order of Contact Theorem proved in [9], 3.5, for embedded varieties with arbitrary dimension). See [12] for a very useful survey. For related details, see the proof of Proposition 3.

Notice that if $\mu = 0$, then (1) gives $\deg(X) \equiv 0 \mod p$.

Remark 1. Take the set-up of Definition 1.

(a) Since a strange curve (not a line) has a unique strange point, the point O is uniquely determined by X. Hence we do not need to specify it to check if a strange curve is flat or not.

(b) The assumption $(\mu, s) = (0, 1)$ implies that $\ell_O|X$ is generically injective. Flatness implies that $\ell_O|X$ is injective, but it is far stronger. We have $r_X(O) \ge 2$ if and only if $O \notin X$. We have $r_X(O) \ge 3$ if and only if $O \notin X$ and $\ell_O|X$ injective. If $\mu = 0$, then the flatness of a strange curve is equivalent to $r_X(O) = n+1$ (use that $r_X(P) \le n+1$ for any $P \in \mathbb{P}^n$ and any non-degenerate reduced subset $X \subset \mathbb{P}^n$ and that for any finite $S \subset X$ we have $\dim(\langle \ell_O(S)\rangle) < \dim(\langle S\rangle)$ if and only if $O \in \langle S\rangle$).

(c) Part (b) shows that the "if" part of the following theorem is just the definition of flatness of a strange curve. It also gives the "only if" part if we first prove that X is a strange point of X with invariants $(\mu, s) = (0, 1)$.

Theorem 2. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve and $P \in \mathbb{P}^n$. We have $r_X(P) \ge n+1$ (i.e. $r_X(P) = n+1$) if and only if X is a flat strange curve and P is the strange point of X.*

V. Bayer and A. Hefez gave explicit equations for all plane strange curves in terms of the invariants μ, s and p^e introduced in Definition 1 ([4]). Later we extended the construction to strange varieties with a fixed strange point O, fix integers μ, s, p^e and a fixed image $T \subset \mathbb{P}^{n-1}$ with respect to the linear projection from O ([3]). All strange curves X such that $O \notin X$, $s = 1$ and $\ell_O(X)$ is a rational normal curve (where O is the strange point of X) are flat (Proposition 2). These curves are explicitely described by one equation in a Hirzebruch surface F_{n-1} ([3]). The other flat strange curves are very strange (Proposition 1) and we know only one example of these flat curves (see Example 1, i.e. [18], Example 1.2). See Remark 2 for another reason to say that the flat curves X with $\ell_O(X)$ a rational normal curve are "almost maximally linearly independent from the set-theoretic point of view".

The topic considered in [15] is very active (see also [7], [6], [5] and references therein). We stress that [15] and the other quoted papers are over $\mathbb{C}$: none of their statements and proofs is affected by the examples given here.

2. Proofs and related results

Proof of Theorem 1. If $P \in X$, then $r_X(P) = 1$. Hence to prove parts (a) and (b) we may assume $P \notin X$. First assume $m = 1$. Assume $r_X(P) \ge n+1$. Hence for a general hyperplane H containing P the set $(X \cap H)_{red}$ does not span H. Since X is connected, the cohomology exact sequence of the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H}(1) \to 0$$

gives that the scheme $X \cap H$ spans H. Thus $X \cap H$ is not reduced. Since $P \notin X$ and H is general among the hyperplanes containing P, $H \cap \mathrm{Sing}(X) = \emptyset$. Hence the non-reducedness of $X \cap H$ and the generality of H implies that X is a strange curve with P as its strange point. In the case $m = 1$ we have $r_X(P) \leq n+1$ for all P, because X spans $\mathbb{P}^n$ proving parts (a) and (b) in the case $m = 1$.

Now assume $m \geq 2$ and that Theorem 1 is true for varieties of dimension $m-1$. Assume the existence of $P \in \mathbb{P}^n$ such that $r_X(P) \geq n+2-m$, but P is not a strange point of X. Fix a general hyperplane H containing P. Let $\ell_P : \mathbb{P}^n \backslash \{P\} \to \mathbb{P}^{n-1}$ be the linear projection from P. Since $P \notin X$, $\ell_P|X$ is a finite morphism. Bertini's theorem gives that $X \cap H$ is geometrically integral ([11], part 4) of Th. I.6.3). Fix a general $Q \in (X \cap H)_{reg}$. For general H we may take as Q a general point of X. Hence $P \notin T_Q X$. Hence $P \notin (T_Q X) \cap H = T_Q(X \cap H)$. Thus P is not a strange point of $X \cap H$. The inductive assumption gives $r_{X \cap H}(P) \leq (n-1)-(m-1)+1 = n-m+1$. Since $r_X(P) \leq r_{X\cap H}(P)$, we proved part (a) for all m, X, P.

Now assume that P is a strange point of X. Since we proved part (b) in the case $m = 1$, we may assume $m \geq 2$. Fix an integer $k \geq 3$ and a general $Q \in X_{reg}$. Let Y be the intersection of X with a general degree k hypersurface W such that $Q \in W$. The scheme $Y \backslash \{Q\}$ is geometrically integral by the characteristic free version of Bertini's theorem for very ample linear systems on non-complete varieties ([11], part 4) of Th. I.6.3). Since $k \geq 3$, it is easy to find W such that $Y = X \cap W$ is smooth at Q. Hence Y is geometrically integral and $Q \in Y_{reg}$. Since $k \geq 3$, we may find W as above such that $P \notin T_Q W$. Hence $P \notin T_Q W \cap T_Q X = T_Q Y$. Hence P is not a strange point of Y. Part (a) applied to Y gives $r_X(P) \leq r_Y(P) \leq n-(m-1)+1$. □

Proof of Theorem 2. By part (c) of Remark 1 it is sufficient to prove the "only if" part. Fix X, P such that $r_X(P) \geq n+1$. The case $m = 1$ of Theorem 1 implies $r_X(P) = n+1$ and that P is a strange point of X. Call μ, s and p^e the invariants of X with respect to the linear projection ℓ_P from P. Since $r_X(P) \geq 2$, $P \notin X$, i.e. $\mu = 0$. Notice that $s = 1$ if and only if $\ell_P|X$ has separablle degree 1, i.e. it is generically injective. Since $r_X(P) \geq 3$, we have $\sharp((X \cap D)_{red}) \leq 1$ for every line D such that $P \in D$. Thus $\ell_P|X$ in injective. Thus $s = 1$. As observed in part (c) of Remark 1 if $(\mu, s) = (0, 1)$ and P is the strange point of X, then the definition of flatness is equivalent to $r_X(P) \geq n+1$. □

Proposition 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a non-degenerate and flat strange curve with O as its strange point. Then either X is very strange or $\ell_O(X)$ is a rational normal curve.*

Proof. Let O be the strange point of X. Set $d := \deg(\ell_O(X))$. If $d = n-1$, then $\ell_O(X)$ is a rational normal curve. Now assume $d \geq n$. By assumption $\mu = 0$ and $s = 1$. Fix a general $S \subset X$ such that $\sharp(S) = n-1$. Hence $\sharp(\ell_O(S)) = n-1$ and $\ell_O(S)$ spans a hyperplane of $\mathbb{P}^{n-1}$. Since $d \geq n$, there is $U \in \ell_O(X) \backslash \ell_O(S)$ such that $U \in \langle \ell_O(S) \rangle$. Fix $V \in X$ such that $\ell_O(V) = U$. Hence $\sharp(S \cup \{V\}) = n$. Since X is flat, $V \in \langle S \rangle$. Since this is true for a general $S \subset X$ such that $\sharp(S) = n-1$, X satisfies the definition of a very strange curve. □

Proposition 2. *Let $X \subset \mathbb{P}^n$, $n \geq 2$, be a non-degenerate and strange curve with O as its strange point and invariants $\mu = 0$ and $s = 1$, i.e. assume $O \notin X$ and that $\ell_O|X$ is generically injective. If either $n = 2$ or $\ell_O(X)$ is a rational normal curve of $\mathbb{P}^{n-1}$ (i.e. if $\deg(X) = (n-1)p^e$, where p^e is the inseparable degree of $\ell_O|X$), then X is flat.*

Proof. Fix $S \subset X$ such that $\sharp(S) \leq n$. Let $u : C \to X$ be the normalization map. By assumption $\ell_O(X) \cong \mathbb{P}^1$ (even if $n = 2$). Since $\ell_O|X : X \to T \cong \mathbb{P}^1$ is purely inseparable, $C \cong \mathbb{P}^1$. Since $s = 1$, the morphism $\ell_O|X \circ u : \mathbb{P}^1 \to \mathbb{P}^1$ is purely inseparable. Hence it is injective. Thus the morphism $\ell_O|X$ is injective, not just generically injective. Hence $\sharp(\ell_O(S)) = \sharp(S) \le n$. Since any n points of a rational normal curve of $\mathbb{P}^{n-1}$ are linearly independent, we get $\dim(\langle \ell_O(S)\rangle) = \sharp(S)-1$. $\square$

Remark 2. Take X as in Proposition 2. The proof of Proposition 2 gives that every $S \subset X$ such that $\sharp(S) \le n$ is linearly independent, i.e. X has no codimension 2 multisecant linear subspace from the set-theoretical point of view (but of course every tangent line of X at one of its smooth points contains a length p^e subscheme of X). We stress again that all curves X as in Proposition 2 are explicitely constructed in [3]. The rational normal curves of $\mathbb{P}^n$ are the only integral curves for which no hyperplane contains $n+1$ points of the curve, i.e. for which the reduction of every codimension 1 linear section is linearly independent.

Example 1. Here we check that the example of a very strange curve given in [18], Example 1.2, is a flat strange curve. Fix an integer $n \ge 3$, a prime p and a p-power q. Here $q = p^e$ is the inseparable degree of the linear projection from the strange point. Fix homogeneous coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n$ and homogeneous coordinates $x_1, \dots, x_n$ of $\mathbb{P}^{n-1}$. Set $A := (0;\dots;0;1;0)$ and $O := (1;0;\dots;0;0)$. We recall that every point of the vertex of a cone T is a strange point of T. An integral hypersurface $\{f(x_0,\dots,x_n) = 0\}$ has O as one of its strange points if and only if in each monomial of f with a non-zero coefficient the variable x_0 appears with exponent divisible by p. Let X be the scheme with equations $x_0^q - x_1x_n^{q-1}, x_1^q - x_2x_n^{q-1}, \dots, x_{n-2}^q - x_{n-1}x_n^{q-1}$. The point O is a strange points of the $n-1$ hypersurfaces with these equations (the latter $n-2$ hypersurfaces are cones with vertex containing O). Set $X' := X \cap \{x_n \ne 0\}$. We have $(X \cap \{x_n = 0\})_{red} = \{A\}$. Since X is given by $n-1$ equations, each irreducible component of X_{red} has dimension at least 1. Hence A is in the closure of X'. Set $t := x_0/x_n$. The scheme $(X')_{red}$ has a rational parametrization

$$t \mapsto (t, t^q, t^{q^2}, \dots, t^{q^{n-1}}), \tag{2}$$

because in X' we have $x_i/x_n = (x_{i-1}/x_n)^q$ for every $i \in \{1,\dots,n-1\}$. Hence $(X')_{red}$ is integral, smooth, rational and its closure X_{red} in $\mathbb{P}^n$ has O as its strange point. Since $\deg(X_{red}) = q^{n-1}$ and X_{red} is set-theoretically the intersection of $n-1$ hypersurfaces of degree q, the algebraic set X_{red} is the complete intersection of these hypersurfaces, outside finitely many points. Hence the scheme X is a complete intersection and it is reduced outside finitely many points. Since X is a complete intersection, each local ring $\mathcal{O}_{X,Q}$, $Q \in X_{red}$, is Cohen-Macaulay. Hence X has no embedded component and it is generically reduced. Thus it is reduced. We have $O \notin X$. Set $Y := \ell_O(X) \subset \mathbb{P}^{n-1}$, $Y' := Y \cap \{x_n \ne 0\}$ and $A' := (0;\dots;1;0) = \ell_O(A) \in Y$. Since $\ell_O((t;t^q;\dots;t^{q^{n-1}};1)) = (t^q;\dots;t^{q^{n-1}};1)$ for all $t \in \mathbb{K}$, the curve Y' has a parametrization

$$z \mapsto (z, z^q, \dots, z^{q^{n-2}}), \tag{3}$$

where $z = t^q$. Hence $\ell_O|X' : X' \to Y'$ is injective and purely inseparable with inseparable degree q. Thus X has parameters $(\mu, s, p^e) = (0, 1, q)$. The parametrization (3) shows that Y' is smooth, that Y is strange with $O'' := (1;0;\dots;0;0)$ as its strange point and that $Y\backslash Y' = \{A'\}$. Fix linearly independent $P_1, \dots, P_n \in X'$ and set $S := \{P_1,\dots,P_n\}$ and $M := \langle S\rangle$. The parametrization (2) shows that $(M \cap X')_{red} = \{P_1 + a_1(P_2 - P_1) + \cdots + a_{n-1}(P_n - P_1)\}$, where each a_i is an arbitrary element of $\mathbb{F}_q$. Since $\sharp((M \cap X')_{red}) = q^{n-1} = \deg(X)$, we get that this is a scheme-theoretic intersection and that $M \cap (X\backslash X') = \emptyset$. Since $M \cap X = (M \cap X')_{red}$ scheme-theoretically and $O \in T_{P_i}X$, we have $O \notin M$, i.e. $\dim(\ell_O(M)) = n-1$. Recall that $X\backslash X' = \{A\}$. Fix $S_1 \subset X$ such that $\sharp(S_1) = n$, $A \in S_1$ and S_1 is linearly independent. Let M_1 be the hyperplane spanned by S_1. Set $S_2 := S_1\backslash\{A\}$ and write $S_2 := \{P_1, \dots, P_{n-1}\}$. Set $Q_i := \ell_O(P_i)$, $1 \le i \le n-1$. We proved that $\sharp(\ell_O(S_2)) = n-1$, $\ell_O(S_2) \subset Y'$ and that $\ell_O(S_2)$ is linearly independent. Set $M_2 := \langle \ell_O(S_2)\rangle$. Since $A' = \ell_O(A)$, to conclude the proof of the flatness of X it is sufficient to prove $A' \notin M_2$. Let $E \subset \mathbb{P}^{n-1}$ the set $\{Q_1 + a_1(Q_2 - Q_1) + \cdots + a_{n-2}(Q_{n-1} - Q_1)\}$, where each a_i is an arbitrary element of $\mathbb{F}_q$. Since $P_1 + a_1(P_2 - P_1) + \cdots + a_{n-2}(P_{n-1} - P_1) \in X'$ for all $a_i \in \mathbb{F}_q$, we have $E \subseteq M_2 \cap \ell_O(X')$. Since $\ell_O|X'$ is injective, we have $\sharp(E) = q^{n-2} = \deg(Y)$. Thus $E = M_2 \cap Y$ and $(Y\backslash\ell_O(X')) \cap M_2 = \emptyset$. Since $\{A'\} = Y\backslash\ell_O(X')$, we get $A' \notin M_2$. Thus X is flat.

Remark 3. A theorem of Luiss' says that there is a unique smooth strange curve (if we exclude the lines): a smooth plane conic in characteristic 2 ([13], Proposition 3, or [10], Theorem IV.3.9). If $p = 2$ a smooth plane conic is obviously flat. This example shows that if $n = 2$ and $p = 2$ the ranks of the rational normal curves of $\mathbb{P}^n$ are not as in characteristic zero (see [7], [15], 4.1, or [5], 3.1). This phenomenon does not occur when $n = 3$. Let $C \subset \mathbb{P}^3$ be a rational normal curve. Let $TC := \cup_{Q\in C} T_QC \subset \mathbb{P}^3$ denote the tangent developable of C. If $P \in C$, then $r_C(P) = 1$. If $P \notin TC$, then $r_C(P) = 2$, because $\mathbb{P}^3$ is the secant variety of C ([1], Remark 1.6). Fix $P \in TC\backslash C$, say $P \in T_QC\backslash\{Q\}$ with $Q \in C$. Assume $r_C(P) = 2$ and take $P_1, P_2 \in C$ such that $P_1 \ne P_2$ and $P \in \langle\{P_1, P_2\}\rangle$. Since any length 3 scheme $Z \subset C$ spans a plane, $Q \notin \langle\{P_1, P_2\}\rangle$. Since $P \in T_QC \cap \langle\{P_1, P_2\}\rangle$, the linear space $M := \langle T_QC \cup \{P_1, P_2\}\rangle$ is a plane and $\text{length}(M \cap C) \ge 4$. Since $\deg(C) \ge 3$, we get a contradiction. Hence $r_C(P) \ge 3$. Since C is not strange, Theorem 1 gives $r_C(P) = 3$. Hence the stratification by ranks of C is the same as in characteristic zero.

Fix an integer $m \ge 2$. Here we construct m-dimensional examples of pairs (X, P) such that $r_X(P) = n + 2 - m$, i.e. such that the inequality in part (b) of Theorem 1 is an equality. Just taking cones we get an m-dimensional example from any one-dimensional example with the same codimension in an ambient projective space. This is the only example we know of pairs (X, P) with $m \ge 2$ and $r_X(P) = n+2-m$, i.e. a pair for which part (b) of Theorem 1 is sharp. Are there other examples?

Remark 4. Fix integers $n > m \ge 2$, an $(n-m+1)$-dimensional linear subspace M of $\mathbb{P}^n$ and an $(m-2)$-dimensional linear subspace N of $\mathbb{P}^n$ such that $M \cap N = \emptyset$, i.e. a complementary subspace. For any variety $Y \subset M$ let $C(N, Y) \subset \mathbb{P}^n$ denote the cone with vertex N and Y as its basis. Hence for each $O \in M$ the scheme $C(N, O)$ is an $(m-1)$-dimensional linear subspace of $\mathbb{P}^n$. We claim that $r_{C(N,Y)}(P) = r_Y(O)$ for every $P \in C(N, O)\backslash N$. Fix $P \in C(N, O)\backslash N$. Take an $(n-m+1)$-dimensional linear subspace M' of $\mathbb{P}^n$ such that $P \in M'$ and $N \cap M' = \emptyset$. The linear projection from N induces an isomorphism of pairs $(C(N, Y) \cap M', P) \cong (Y, O)$ as pairs of subvarieties, respectively of M' and of M. Thus $r_{C(N,Y)}(P) \le r_{C(N,Y)\cap M'}(P) = r_Y(O)$. To prove the reverse inequality we fix $P \in C(N, O)$ and $S \subset C(N, Y)$ computing $r_{C(N,Y)}(P)$. The image $S' \subset M$ of the linear projection of S from N is a set such that $\sharp(S') \le \sharp(S) = r_{C(N,Y)}(P)$. Since $O \in \langle S'\rangle$, we get $r_Y(O) \le \sharp(S') \le r_{C(N,Y)}(P)$. Taking as Y a flat curve with strange point O, $X = C(N, Y)$ and

$V = C(N, O)$ we get the existence (for all $n > m \geq 2$) of an integral, non-degenerate and m-dimensional variety $X \subset \mathbb{P}^n$ with as set of its strange points an $(m-1)$-dimensional linear space V and $r_X(P) = n - m + 2$ for all $P \in V \backslash N$, where N is an $(m-2)$-dimensional linear space and $N \subset X$.

Proposition 3. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $O \in \mathbb{P}^n$ and assume that O is a strange point of X, but that X is not a cone with vertex containing O. Then* $\deg(X) \geq p \cdot (n-m)$.

Proof. Fix $A \in \mathbb{P}^n \backslash \{O\}$ and take any integral quasi-projective variety $E \subseteq \mathbb{P}^n \backslash \{O\}$ such that $A \in E_{reg}$. Set $x := \dim(E)$. The inclusion $j : E \subseteq \mathbb{P}^n$ induces an inclusion between the abstract tangent spaces $\Theta_{E,A}$ of E at A and the abstract tangent space $\Theta_{\mathbb{P}^n,A}$ of $\mathbb{P}^n$ at A. As usual in projective geometry we "complete" these vector spaces $\Theta_{E,A}$ and $\Theta_{\mathbb{P}^n,A}$ to projective spaces, respectively of dimension x and n, and call them T_AE and $T_A\mathbb{P}^n = \mathbb{P}^n$. Since $A \neq 0$, the submersion $\ell_O : \mathbb{P}^n \backslash \{O\} \to \mathbb{P}^{n-1}$ induces a linear surjective map of $\mathbb{K}$-vector spaces $\rho_O(A) : \Theta_{\mathbb{P}^n,A} \to \Theta_{\mathbb{P}^{n-1},\ell_O(A)}$. Since $\rho_O(A)$ is surjective, its kernel is one-dimensional. If we identify $\Theta_A\mathbb{P}^n$ with an affine n-dimensional open subset of $T_A\mathbb{P}^n = \mathbb{P}^n$, then the closure of this kernel is the line $\langle \{O, A\} \rangle$ (in the case $x = 1$, see [13], lines 3–4 of p. 215). Thus the differential of $\ell_O|E$ at A is injective if and only if $O \notin T_AE$. Thus the differential of $\ell_O|E$ at a general point of E is injective if and only if the closure $\overline{E} \subseteq \mathbb{P}^n$ of E is not strange with O as one of its strange points.

Let $T \subset \mathbb{P}^{n-1}$ denote the closure of $\ell_O(X \backslash \{O\})$. Since X is not a cone with vertex containing O, $\ell_O|X \backslash \{O\}$ is a generically finite morphism. Hence $\dim(T) = m$. Since T spans $\mathbb{P}^{n-1}$, we have $\deg(T) \geq n - m$. Since $\ell_O|X \backslash \{O\}$ is generically finite, the function field $K(X)$ of X is a finite extension of the function field $K(T)$. Since O is a strange point of X, this extension of fields is not separable (use the geometric interpretation of $\rho_O(A)$ just given and the differential criterion of separability, i.e. [17], Theorem 26.6, or [16], Th. 59 at p. 191, quoted in [10], Theorem II.8.6). Call p^e, $e \geq 1$, the inseparable degree of this extension of fields. A general fiber of $\ell_O|X \backslash \{O\}$ is a disjoint union of finitely many connected zero-dimensional schemes, each of them with degree p^e. Hence $\deg(X) \geq p^e \cdot \deg(T) \geq p(n-m)$. □

In the set-up of Proposition 3 if $O \in X$, then $\deg(X) > p \cdot (n-m)$. Proposition 3 is very weak, but we are unable to make a substantial improvement of it. In the case of a strange curve X the formula (1) relates $\deg(X)$ to other data. Nothing more can be said in the one-dimensional case. Indeed, the construction of [3] shows that we may take an arbitrary T spanning $\mathbb{P}^{n-1}$ and then find a solution X with arbitrary $e \geq 1$ and $\mu \geq 0$. Formula (1) is very useful to check if a curve X is strange. We observed after Definition 1 that if $\deg(X)/p \notin \mathbb{Z}$, then either X is not strange or its strange point belongs to X. If X is strange, we also see that the image curve T has much lower degree and hence it should be easier.

It seems to be very difficult to construct very strange curves. We know only the examples given in [18]. We expect that if they exist, then they have very large degree, at least p^{n-1} in $\mathbb{P}^n$.

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Dept. of Mathematics, University of Trento, 38050 Povo (TN), Italy
E-mail address: `ballico@science.unitn.it`